STEP I 2011 Solutions and Mark Scheme

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Contents

Foreword

This document contains model solutions to the 2011 STEP Mathematics Paper I. The solutions are fully worked and contain more detail and explanation than would be expected from candidates. They are intended to help students understand how to answer the questions, and therefore they are encouraged to attempt them first before looking at these model answers.

This document also contains a Mark Scheme. This was used by the markers during the marking process. It is important to remember that the nature of these questions is such that there may be multiple acceptable ways of answering them. As in any examination, the mark scheme was adapted appropriately for these alternative approaches; these adaptations are not recorded here.

The meanings of the marks are as in the standard GCSE and AS/A2 mark schemes:

- M marks for method
- A marks for correct answers, dependent on gaining the corresponding M mark(s). If there is no method shown, but the answer is correct, either the M marks should be awarded along with the A mark or no marks can be awarded. What to do in any particular case should be discussed with the PE.
- B marks are independent accuracy marks
- ft means that incorrect working is followed through
- dep means this method mark is dependent upon gaining the previous method mark
- cao/cso means ‘correct answer/solution only’
- SC means ‘special case’, and applies when the regular mark scheme has given the student (almost) no marks
- ‘condone . . . ’ means ‘award the mark even if the candidate has made the specified error’
- AG means ‘answer given’ in the question; sufficient working has to be demonstrated to show that the candidate has reached the given answer from their work rather than simply copying it from the question

Question 1

> *(i) Show that the gradient of the curve* $\dfrac{a}{x} + \dfrac{b}{y} = 1$*, where* $b \neq 0$*, is* $-\dfrac{ay^2}{bx^2}$.

We begin by differentiating the equation of the curve ($ax^{-1} + by^{-1} = 1$) implicitly with respect to x, to get

$$-ax^{-2} - by^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$-\frac{b}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2},$$

giving our desired result

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ay^2}{bx^2}.$$

An alternative, but more complicated method, is to rearrange the equation first to get y in terms of x before differentiating. We have, on multiplying by xy,

$$ay + bx = xy, \tag{1}$$

so that $(x-a)y = bx$, which gives

$$y = \frac{bx}{x-a}.$$

We can now differentiate this using the quotient rule to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x-a) - bx.1}{(x-a)^2} = \frac{-ab}{(x-a)^2}.$$

The challenge is now to rewrite this in the form required. We can rearrange equation (1) to get $(x-a)y = bx$, so that $(x-a) = bx/y$. Substituting this into our expression for the derivative then gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ab}{(bx/y)^2} = -\frac{aby^2}{b^2x^2} = -\frac{ay^2}{bx^2}$$

as required.

Marks

M1: Implicitly differentiating equation
A1: $-ax^{-2}$ term in derivative
A1: $-by^{-2}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ term in derivative
A1 cso (AG): Rearranging to find $\mathrm{d}y/\mathrm{d}x$

Alternative method: Rearranging equation
M1: Rearranging equation to get $y = \cdots$
A1: Reaching $y = b(1 - a/x)^{-1} = bx/(x - a)$ or equivalent
M1: Differentiating this to get $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-ab}{(x-a)^2}$
A1 (AG): Substituting to deduce that this equals $-ay^2/bx^2$
[Total for this first part: 4 marks]

> *The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.*

Rearranging the equation of the straight line $ax + by = 1$ as $y = -(\frac{a}{b})x + \frac{1}{b}$ shows that its gradient is $-a/b$.

Then using the above result for the gradient of the curve, we require that

$$-\frac{aq^2}{bp^2} = -\frac{a}{b},$$

so $q^2/p^2 = 1$, that is $p^2 = q^2$ or $p = \pm q$.

Marks

B1: Gradient of straight line
M1: Equating gradients of line and curve at (p, q)
A1 cso: Deducing given result
[Total for this part: 3 marks]

> *Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.*

Since (p, q) lies on both the straight line and the curve, it must satisfy both equations, so

$$ap + bq = 1 \qquad \text{and} \qquad \frac{a}{p} + \frac{b}{q} = 1.$$

Now if $p = q$, then the first equation gives $(a + b)p = 1$ and the second gives $(a + b)/p = 1$, and multiplying these gives $(a + b)^2 = 1$.

Alternatively, if $p = -q$, then the first equation gives $(a - b)p = 1$ and the second equation gives $(a - b)/p = 1$, and multiplying these now gives $(a - b)^2 = 1$.

Marks

B1: Substituting (p, q) into equations for line and curve (can be awarded earlier if seen; will often be joined with the next step/mark)

M1: Substituting $p = \pm q$ or $q = \pm p$ into both equations

M1: Factorising or some other useful step towards required result

A1: Multiplying equations or equivalent to reach either $(a + b)^2 = 1$ or $(a - b)^2 = 1$

A1: Repeating the above for the other case

[Total for this part: 5 marks; Total for part (i): 12 marks]

(ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.

We can find the derivative of this curve as above. A slick alternative is to notice that this is identical to the above curve, but with b replaced by $-b$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ay^2}{bx^2}.$$

The gradient of the straight line is $-a/b$ as before, so as this line is normal to the curve at the point (p, q), say, we have

$$\frac{aq^2}{bp^2}\left(-\frac{a}{b}\right) = -1$$

as perpendicular gradients multiply to -1; thus $a^2q^2/b^2p^2 = 1$, or $a^2q^2 = b^2p^2$.

We therefore deduce that $aq = \pm bp$, which we can divide by $pq \neq 0$ to get $\frac{a}{p} = \pm\frac{b}{q}$.

Now since (p, q) lies on both the straight line and the curve, we have, as before,

$$ap + bq = 1 \qquad \text{and} \qquad \frac{a}{p} - \frac{b}{q} = 1.$$

Now if $\frac{a}{p} = \frac{b}{q}$, the second equation would become $0 = 1$, which is impossible. So we must have $\frac{a}{p} = -\frac{b}{q}$, giving

$$\frac{a}{p} + \frac{a}{p} = 1,$$

so that $\frac{a}{p} = -\frac{b}{q} = \frac{1}{2}$, giving $p = 2a$ and $q = -2b$.

Substituting this into the equation of the straight line yields

$$a.2a + b.(-2b) = 1,$$

so that $a^2 - b^2 = \frac{1}{2}$ as required.

Marks

B1: Gradient of curve

M1: Product of normal gradients is -1

A1: Reaching $a^2q^2/b^2p^2 = 1$; following through an incorrect gradient of $-ay^2/bx^2$ to get $a^2q^2/b^2p^2 = -1$ is awarded this mark (but further progression is impossible)

A1: Reaching $aq = \pm bp$

M1: Rejecting $aq = +bp$ possibility

M1: Substituting to get $2a/p = 1$ or $2b/q = -1$
A1: Reaching $p = 2a$ and $q = -2b$ or equivalent
A1 cso AG: Substituting to get $a^2 - b^2 = \frac{1}{2}$
[Total for part (ii): 8 marks]

Question 2

The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x\,.$

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ *in terms of* e *and* E.

Approach 1: Using polynomial division

Using polynomial division or similar, we find that we can write

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Therefore our first integral becomes

$$\begin{aligned}
\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \int_0^1 \Big(1 - \frac{1}{1+x}\Big)\mathrm{e}^x\,\mathrm{d}x \\
&= \int_0^1 \mathrm{e}^x\,\mathrm{d}x - \int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x \\
&= \big[\mathrm{e}^x\big]_0^1 - E \\
&= \mathrm{e} - 1 - E,
\end{aligned}$$

as required.

We can play the same trick with the second integral, as

$$\frac{x^2}{1+x} = x - 1 + \frac{1}{1+x},$$

so that

$$\begin{aligned}
\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \int_0^1 \Big(x - 1 + \frac{1}{1+x}\Big)\mathrm{e}^x\,\mathrm{d}x \\
&= \int_0^1 x\mathrm{e}^x\,\mathrm{d}x - \int_0^1 \mathrm{e}^x\,\mathrm{d}x + \int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x.
\end{aligned}$$

Now we can use integration by parts for the first integral to get

$$\begin{aligned}
\int_0^1 x\mathrm{e}^x\,\mathrm{d}x &= \big[x\mathrm{e}^x\big]_0^1 - \int_0^1 \mathrm{e}^x\,\mathrm{d}x \\
&= \mathrm{e} - (\mathrm{e} - 1) \\
&= 1.
\end{aligned}$$

Therefore

$$\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x = 1 - (\mathrm{e} - 1) + E = 2 - \mathrm{e} + E.$$

Approach 2: Substitution

We can substitute $u = 1 + x$ to simplify the denominator in the integral. This gives us

$$\int_0^1 \frac{xe^x}{1+x}\,dx = \int_1^2 \frac{(u-1)e^{u-1}}{u}\,du$$
$$= \int_1^2 e^{u-1} - \frac{e^{u-1}}{u}\,du.$$

The first part of this integral can be easily dealt with. The second part needs the reverse substitution to be applied, replacing u by $1 + x$, giving

$$\left[e^{u-1}\right]_1^2 - \frac{e^x}{1+x}\,dx = e - 1 - E.$$

This is essentially identical to the first approach. The second integral follows in the same way.

Approach 3: Integration by parts

Integration by parts is trickier for this integral, as it is not obvious how to break up our integral. We use the parts formula as written in the formula booklet: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

There are several ways which work (and many which do not). Here is a relatively straightforward approach. For the first integral, we take

$$u = \frac{x}{1+x} \qquad \text{and} \qquad \frac{dv}{dx} = e^x$$

so that

$$\frac{du}{dx} = \frac{1}{(1+x)^2} \qquad \text{and} \qquad v = e^x.$$

We then get

$$\int_0^1 \frac{xe^x}{1+x}\,dx = \left[\frac{xe^x}{1+x}\right]_0^1 - \int_0^1 \frac{e^x}{(1+x)^2}\,dx$$
$$= \tfrac{1}{2}e - \int_0^1 \frac{e^x}{(1+x)^2}\,dx.$$

The difficulty is now integrating the remaining integral. We again use parts, this time taking

$$u = e^x \qquad \text{and} \qquad \frac{dv}{dx} = \frac{1}{(1+x)^2}$$

so that

$$\frac{du}{dx} = e^x \qquad \text{and} \qquad v = -\frac{1}{1+x}.$$

This gives

$$\int_0^1 \frac{e^x}{(1+x)^2}\,dx = \left[-\frac{e^x}{1+x}\right]_0^1 - \int_0^1 -\frac{e^x}{1+x}\,dx$$
$$= -\tfrac{1}{2}e + 1 + E.$$

Combining this result with the first result then gives

$$\int_0^1 \frac{xe^x}{1+x}\,dx = \tfrac{1}{2}e - (-\tfrac{1}{2}e + 1 + E) = e - 1 - E.$$

For the second integral, we use a similar procedure, this time taking

$$u = \frac{x^2}{1+x} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x+x^2}{(1+x)^2} \qquad \text{and} \qquad v = \mathrm{e}^x.$$

We then get

$$\begin{aligned}\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \left[\frac{x^2\mathrm{e}^x}{1+x}\right]_0^1 - \int_0^1 \frac{(2x+x^2)\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\mathrm{e} - \int_0^1 \frac{(2x+x^2)\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$

The integral in the last step can be handled in several ways; the easiest is to write

$$\frac{2x+x^2}{(1+x)^2} = \frac{x^2+2x+1-1}{(1+x)^2} = 1 - \frac{1}{(1+x)^2}$$

and then use the earlier calculation of $\int \mathrm{e}^x/(1+x)^2\,\mathrm{d}x$ to get

$$\begin{aligned}\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \tfrac{1}{2}\mathrm{e} - \left(\int_0^1 \mathrm{e}^x - \frac{\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x\right) \\ &= \tfrac{1}{2}\mathrm{e} - [\mathrm{e}^x]_0^1 + (-\tfrac{1}{2}\mathrm{e} + 1 + E) \\ &= -\mathrm{e} + 1 + 1 + E \\ &= 2 - \mathrm{e} + E.\end{aligned}$$

Marks

Approaches 1 and 2:

M1: Writing $x/(1+x)$ as $1 - 1/(1+x)$ or substituting $u = 1+x$ and breaking up the integral

M1: Splitting integral and integrating $\int \mathrm{e}^x\,\mathrm{d}x$ correctly, including substituting back $x = u-1$ if relevant

A1 cso AG: Reaching given answer

Approach 3 (parts):

M2: Applying parts correctly twice (once is insufficient to get anywhere)

A1 cso AG: Reaching given answer

[Total: 3 marks for first integral]

Approaches 1 and 2:

M1: Writing $x^2/(x+1)$ as $x - x/(x+1)$ or as $x - 1 + 1/(x+1)$, or substituting $u = 1+x$ and breaking up the integral

M1: Using parts for $\int x\mathrm{e}^x\,\mathrm{d}x$

A1: Correct $\int x\mathrm{e}^x\,\mathrm{d}x$, condoning at most one arithmetic error (e.g., evaluating $x\mathrm{e}^x$ as 1 when $x = 0$) but not a sign error in the parts formula

M1: Using earlier results to evaluate everything else, including substituting $x = u-1$ if relevant

A1 cao: Correctly evaluating integral to get $2 - \mathrm{e} + E$, or unsimplified equivalent.

Approach 3 (parts):

M1: Useful application of parts – must be "going somewhere" to get any method marks

M1: Either applying parts a second time or using some other technique to make useful progress on the integral

A1: If relevant, correct $\int xe^x\,dx$, condoning at most one arithmetic error (e.g., evaluating xe^x as 1 when $x = 0$) but not a sign error in the parts formula; otherwise correct application of parts

M1: Using earlier results to evaluate everything else

A1 cao: Correctly evaluating integral to get $2 - e + E$, or unsimplified equivalent.

[Total: 5 for second integral, 8 in total for this part of question]

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{e^{\frac{1-x}{1+x}}}{1+x}\,dx$;

This integral looks to be of a vaguely similar form, but with a more complicated exponential part. We therefore try the substitution $u = \frac{1-x}{1+x}$ and see what we get.

If $u = \dfrac{1-x}{1+x}$, then

$$\frac{du}{dx} = \frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2},$$

so that $\frac{dx}{du} = -\frac{1}{2}(1+x)^2$. Also, when $x = 0$, $u = 1$, and when $x = 1$, $u = 0$.

We can also rearrange $u = \dfrac{1-x}{1+x}$ to get

$$\begin{aligned} (1+x)u &= 1-x \\ \text{so} \qquad ux + x &= 1-u \\ \text{giving} \qquad x &= \frac{1-u}{1+u}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 \frac{e^{\frac{1-x}{1+x}}}{1+x}\,dx &= \int_1^0 \frac{e^u}{1+x}\left(-\tfrac{1}{2}(1+x)^2\right)du \\ &= \int_0^1 \tfrac{1}{2}e^u(1+x)\,du && \text{reversing the limits} \\ &= \int_0^1 \tfrac{1}{2}e^u\left(1+\frac{1-u}{1+u}\right)du \\ &= \int_0^1 \tfrac{1}{2}e^u\left(\frac{2}{1+u}\right)du \\ &= \int_0^1 \frac{e^u}{1+u}\,du \\ &= E. \end{aligned}$$

Marks

M1: Attempting to use substitution $u = \frac{1-x}{1+x}$

M1: Calculating du/dx or dx/du ...

A1: ... correctly

M1: Rearranging $u = \frac{1-x}{1+x}$ to make x the subject

M1: Correctly performing substitution; condone forgetting to change the limits for this mark

A1 cao: Deducing correct integral in terms of u alone; must have evidence of change of limits for this mark

A1: Evaluating integral to E (only follow through earlier sign error or arithmetic slip)

[Total for part (i): 7 marks]

Evaluate also in terms of E and e as appropriate:

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x$

Again we have a different exponent, so we try substituting $u = x^2$, so that $x = \sqrt{u}$, while $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and so $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2x}$. Also the limits $x = 1$ and $x = \sqrt{2}$ become $u = 1$ and $u = 2$, giving us

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{e}^u}{x}\,\frac{1}{2x}\,\mathrm{d}u \\ &= \int_1^2 \frac{\mathrm{e}^u}{2u}\,\mathrm{d}u.\end{aligned}$$

This is very similar to what we are looking for, except that it has the wrong limits and a denominator of $2u$ rather than $u+1$ or perhaps $2(u+1)$. So we make a further substitution: $u = v+1$, so that $v = u-1$ and $\mathrm{d}u/\mathrm{d}v = 1$, giving us

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{e}^u}{2u}\,\mathrm{d}u \\ &= \int_0^1 \frac{\mathrm{e}^{v+1}}{2(v+1)}\,\mathrm{d}v \\ &= \frac{\mathrm{e}}{2}\int_0^1 \frac{\mathrm{e}^v}{v+1}\,\mathrm{d}v \\ &= \frac{\mathrm{e}E}{2},\end{aligned}$$

where on the penultimate line we have written $\mathrm{e}^{v+1} = \mathrm{e}.\mathrm{e}^v$ and so taken out a factor of $\mathrm{e}/2$.

It is also possible to evaluate this integral more directly by substituting $u = x^2 - 1$, so that $x^2 = u + 1$. The details are left to the reader.

Marks

M1: Substituting $u = x^2$, including correctly using the derivative, but condoning wrong limits, ...

A1: ... and getting it all correct

M1 dep: Substituting $u = v + 1$, as before, ...

A1: ... and getting it all correct

A1 cao: Final integral evaluation

Alternative if substitute $u = x^2 - 1$ directly

M1: Attempt to substitute $u = x^2 - 1$ or equivalent

M1: Correctly using derivative and attempting to change limits

A1: All correct

M1: Taking out factor of e

A1 cao: Correct answer

[Total for part (ii): 5 marks]

Question 3

Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

We make use of two of the factor formulæ:

$$\begin{aligned}2\sin A\sin B&=\cos(A-B)-\cos(A+B)\\2\sin A\cos B&=\sin(A+B)+\sin(A-B)\end{aligned}$$

(These can be derived by expanding the right hand sides using the addition formulæ, and then collecting like terms.)

Then initially taking $A=\frac{1}{3}\pi-\theta$ and $B=\frac{1}{3}\pi+\theta$ and using the first of the factor formulæ gives

$$\begin{aligned}4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)&=2\sin\theta\big(\cos(-2\theta)-\cos(\tfrac{2}{3}\pi)\big)\\&=2\sin\theta(\cos 2\theta+\tfrac{1}{2})\\&=2\sin\theta\cos 2\theta+\sin\theta.\end{aligned}$$

We now use the second factor formula with $A=\theta$ and $B=2\theta$ to simplify this last expression to

$$\big(\sin 3\theta+\sin(-\theta)\big)+\sin\theta=\sin 3\theta,$$

as required.

An alternative approach is to expand the second and third terms on the left hand side using the addition formulæ, giving:

$$\begin{aligned}&4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)\\&\quad=4\sin\theta(\sin\tfrac{1}{3}\pi\cos\theta-\cos\tfrac{1}{3}\pi\sin\theta)(\sin\tfrac{1}{3}\pi\cos\theta+\cos\tfrac{1}{3}\pi\sin\theta)\\&\quad=4\sin\theta\big(\tfrac{\sqrt{3}}{2}\cos\theta-\tfrac{1}{2}\sin\theta\big)\big(\tfrac{\sqrt{3}}{2}\cos\theta+\tfrac{1}{2}\sin\theta\big)\\&\quad=4\sin\theta(\tfrac{3}{4}\cos^2\theta-\tfrac{1}{4}\sin^2\theta)\\&\quad=3\sin\theta\cos^2\theta-\sin^3\theta,\end{aligned}$$

while

$$\begin{aligned}\sin 3\theta&=\sin(2\theta+\theta)\\&=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\&=2\sin\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)\sin\theta\\&=3\sin\theta\cos^2\theta-\sin^3\theta.\end{aligned}$$

Thus the required identity holds.

Marks

Factor formula approach:

M1: Using a factor formula once to replace two of the sines with a product

A1: Applying the factor formula correctly and simplifying if relevant

M1: Applying another factor formula to simplify the remaining product(s)

A1 cso (AG): Reaching the stated result

Addition formula approach:
M1: Using addition formulæ twice to expand the two sine terms
A1: Expanding and simplifying to reach a simple expression for the left hand side
M1: Applying the addition and double-angle formulæ or de Moivre's theorem to the right hand side to reach an expression for the right hand side in terms of θ
A1 cso (AG): Reaching the stated result
[Total for this part: 4 marks]

(i) *By differentiating* $(*)$, *or otherwise, show that*

$$\cot \tfrac{1}{9}\pi - \cot \tfrac{2}{9}\pi + \cot \tfrac{4}{9}\pi = \sqrt{3}\,.$$

We can differentiate a product of several terms using the product rule repeatedly. In general, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(uvwt\ldots) = \frac{\mathrm{d}u}{\mathrm{d}x}vwt\ldots + u\frac{\mathrm{d}v}{\mathrm{d}x}wt\ldots + uv\frac{\mathrm{d}w}{\mathrm{d}x}t\ldots + \cdots$$

In our case, we are differentiating a product of three terms, and we get

$$4\cos\theta\sin(\tfrac{1}{3}\pi - \theta)\sin(\tfrac{1}{3}\pi + \theta) - 4\sin\theta\cos(\tfrac{1}{3}\pi - \theta)\sin(\tfrac{1}{3}\pi + \theta) + 4\sin\theta\sin(\tfrac{1}{3}\pi - \theta)\cos(\tfrac{1}{3}\pi + \theta) = 3\cos 3\theta.$$

Now we are aiming to get an expressing involving cot, so we divide this result by $(*)$ to get

$$\cot\theta - \cot(\tfrac{1}{3}\pi - \theta) + \cot(\tfrac{1}{3}\pi + \theta) = 3\cot 3\theta.$$

We now let $\theta = \frac{1}{9}\pi$ to get

$$\cot \tfrac{1}{9}\pi - \cot \tfrac{2}{9}\pi + \cot \tfrac{4}{9}\pi = 3\cot \tfrac{1}{3}\pi = 3/\sqrt{3} = \sqrt{3},$$

and we are done.

Marks
M1: Differentiating $(*)$ using the product rule twice or using the generalised product rule
A1: Correct derivative of $(*)$
M1: Dividing derivative by $(*)$
M1 dep: Substituting $\theta = \frac{1}{9}\pi$
A1 cso (AG): Reaching given result

Alternative approach:
M1: Taking the natural logarithm of $(*)$
A1: Correct derivative of logarithm
M1: Simplifying result to get in terms of cot
M1 A1 as above
[Total for part (i): 5 marks]

(ii) *By setting $\theta = \frac{1}{6}\pi - \phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that*

$$\cot\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) = \cot 3\theta\,.$$

Setting $\theta = \frac{1}{6}\pi - \phi$ in $(*)$ as instructed gives

$$4\sin(\tfrac{1}{6}\pi - \phi)\sin(\tfrac{1}{6}\pi + \phi)\sin(\tfrac{1}{2}\pi - \phi) = \sin 3(\tfrac{1}{6}\pi - \phi).$$

To get cosines from this expression, we will need to use the identity $\sin(\frac{1}{2}\pi - x) = \cos x$. So we rewrite this as

$$4\sin\bigl(\tfrac{1}{2}\pi - (\tfrac{1}{3}\pi + \phi)\bigr)\sin\bigl(\tfrac{1}{2}\pi - (\tfrac{1}{3}\pi - \phi)\bigr)\sin(\tfrac{1}{2}\pi - \phi) = \sin(\tfrac{1}{2}\pi - 3\phi)$$

which allows us to apply our identity to get

$$4\cos(\tfrac{1}{3}\pi + \phi)\cos(\tfrac{1}{3}\pi - \phi)\cos\phi = \cos 3\phi,$$

which is a similar identity for $\cos 3\phi$. Replacing ϕ by θ and reordering the terms in the product gives

$$4\cos\theta\cos(\tfrac{1}{3}\pi - \theta)\cos(\tfrac{1}{3}\pi + \theta) = \cos 3\theta.$$

Now dividing this identity by $(*)$ gives our desired identity for cot:

$$\cot\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) = \cot 3\theta. \tag{$\dagger$}$$

(Note that there is no factor of 4 in this expression.)

Marks

M1: Substituting $\theta = \frac{1}{6}\pi - \phi$ and simplifying the result

M1 dep: Using $\sin(\frac{1}{2}\pi - x) = \cos x$ on this expression at least once, or expanding with addition formulæ

A1 cao: Formula for $\cos 3\theta$ or equivalent in the required form; allow $\cos 3\phi = 4\sin(\frac{1}{6}\pi - \phi)\ldots$ as this is "similar" to the original identity

M1: Dividing by $(*)$

A1 cso (AG): Reaching given identity

[Total for this sub-part: 5 marks]

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi - \operatorname{cosec}\tfrac{5}{9}\pi + \operatorname{cosec}\tfrac{7}{9}\pi = 2\sqrt{3}\,.$$

As before, we differentiate the expression $(\dagger)$ which we have just derived to get

$$\begin{aligned}-\operatorname{cosec}^2\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) &+ \\ \cot\theta\operatorname{cosec}^2(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) &- \\ \cot\theta\cot(\tfrac{1}{3}\pi - \theta)\operatorname{cosec}^2(\tfrac{1}{3}\pi + \theta) &= -3\operatorname{cosec}^2 3\theta.\end{aligned}$$

When we negate this identity and then divide it by $(\dagger)$, we will have lots of cancellation and we will be left with terms of the form $\operatorname{cosec}^2 x/\cot x$. Now

$$\frac{\operatorname{cosec}^2 x}{\cot x} = \frac{1}{\sin^2 x}.\frac{\sin x}{\cos x} = \frac{1}{\sin x\cos x} = \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x,$$

so that the division gives us

$$2\operatorname{cosec} 2\theta - 2\operatorname{cosec} 2(\tfrac{1}{3}\pi - \theta) + 2\operatorname{cosec} 2(\tfrac{1}{3}\pi + \theta) = 6\operatorname{cosec} 6\theta.$$

To get the requested equality, we halve this identity and set $2\theta = \frac{1}{9}\pi$ so that

$$\operatorname{cosec} \tfrac{1}{9}\pi - \operatorname{cosec} \tfrac{5}{9}\pi + \operatorname{cosec} \tfrac{7}{9}\pi = 3\operatorname{cosec} \tfrac{1}{3}\pi = 3.\tfrac{2}{\sqrt{3}} = 2\sqrt{3}$$

as required.

Marks

M1: Differentiating (†) using the product rule twice or the generalised product rule

M1 dep: Dividing by (†) and simplifying terms at least as far as $1/\sin\theta\cos\theta$ or $\tan\theta\operatorname{cosec}^2\theta$

M1 dep on previous M mark: Reaching expressions of the form $2\operatorname{cosec} 2\theta$ (this could be performed later)

A1: Correct identity for 2θ (possibly divided by 2 or negated, possibly with 2θ replaced by θ)

M1 (dependent upon previous M1 being awarded at some point): Substituting $2\theta = \frac{1}{9}\pi$ in expression involving $\operatorname{cosec} 2\theta$ (award if substitute into $1/\sin\theta\cos\theta$ terms and then manipulate to reach $\operatorname{cosec} 2\theta$ terms)

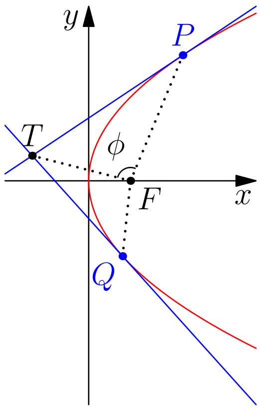
A1 cso (AG): Reaching given equation

[Total for this sub-part: 6 marks]

Question 4

The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $\big(apq, a(p+q)\big)$. You may assume that $p \neq 0$ and $q \neq 0$.

We begin by sketching the graph (though this may be helpful, it is not required):



The equation of the curve is $y^2 = 4ax$, so we can find the gradient of the curve by implicit differentiation:

$$2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4a,$$

and thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y},$$

as long as $y \neq 0$. (Alternatively, we could write $x = y^2/4a$ and then work out $\mathrm{d}x/\mathrm{d}y = 2y/4a$; taking reciprocals then gives us the same result.)

Therefore the tangent at the point P with coordinates $(ap^2, 2ap)$ has equation

$$y - 2ap = \frac{2a}{2ap}(x - ap^2),$$

which can easily be rearranged to give

$$x - py + ap^2 = 0.$$

Since $y = 0$ would require $p = 0$, we can ignore this case, as we are assuming that $p \neq 0$. [In fact, if $y = p = 0$, we can look at the reciprocal of the gradient, $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2a}$, and this is zero, so the line is vertical. In this case, our equation gives $x = 0$, which is, indeed, a vertical line, so our equation works even when $p = 0$.]

Thus the tangent through P has equation $x - py + ap^2 = 0$ and the tangent through Q has equation $x - qy + aq^2 = 0$ likewise.

We solve these equations simultaneously to find the coordinates of T. Subtracting them gives

$$(p - q)y - a(p^2 - q^2) = 0.$$

Since $p \neq q$, we can divide by $p - q$ to get

$$y - a(p+q) = 0,$$

so $y = a(p+q)$ and therefore $x = py - ap^2 = ap(p+q) - ap^2 = apq$.

Thus T has coordinates $\bigl(apq, a(p+q)\bigr)$, as wanted.

Marks

M1: Implicit differentiation, finding $\mathrm{d}x/\mathrm{d}y$ and taking reciprocals, or rearranging to get $y = \pm\sqrt{4ax}$ and differentiating that

A1 cao: Formula for $\mathrm{d}y/\mathrm{d}x$. Note: only award M1 A0 if rearrange to get $y = \sqrt{4ax}$ but do not take account of $\pm$

A1 ft: Equation of tangent through P or Q (does not need to be simplified), either in cartesian or vector form

B1 ft: Equation of tangent through other point (follow through equation of first tangent found)

M1: Solving equations simultaneously to find coordinates of T (reasonable attempt)

B1: Justification for dividing by $p - q$ (sufficient to say that $p \neq q$ or $p - q \neq 0$)

A1 cso (AG): Correct x or y coordinate as given

A1 cso (AG): Reaching given coordinates of T

[Total for this part: 8 marks]

The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that

$$\cos\phi = \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

In the triangle TFP, we can use the cosine rule to find $\cos\phi$:

$$TP^2 = TF^2 + PF^2 - 2.TF.PF.\cos\phi,$$

so that

$$\cos\phi = \frac{TF^2 + PF^2 - TP^2}{2.TF.PF}.$$

Now using Pythagoras to find the distance between two points given their coordinates, we obtain

$$\begin{aligned}
TF^2 &= \bigl(a(pq-1)\bigr)^2 + \bigl(a(p+q)\bigr)^2 \\
&= a^2(p^2q^2 - 2pq + 1 + p^2 + 2pq + q^2) \\
&= a^2(p^2q^2 + p^2 + q^2 + 1) \\
&= a^2(p^2+1)(q^2+1) \\
FP^2 &= \bigl(a(p^2-1)\bigr)^2 + (2ap)^2 \\
&= a^2(p^4 - 2p^2 + 1 + 4p^2) \\
&= a^2(p^4 + 2p^2 + 1) \\
&= a^2(p^2+1)^2 \\
TP^2 &= \bigl(a(pq-p^2)\bigr)^2 + \bigl(a(p+q-2p)\bigr)^2 \\
&= \bigl(ap(q-p)\bigr)^2 + \bigl(a(q-p)\bigr)^2 \\
&= a^2(p^2+1)(q-p)^2
\end{aligned}$$

Thus

$$\begin{aligned} TF^2 + FP^2 - TP^2 &= a^2(p^2+1)(q^2+1+p^2+1-q^2+2pq-p^2) \\ &= 2a^2(1+p^2)(1+pq) \end{aligned}$$

so that

$$\begin{aligned} \cos\phi &= \frac{2a^2(1+p^2)(1+pq)}{2a^2\sqrt{(p^2+1)(q^2+1)(p^2+1)^2}} \\ &= \frac{1+pq}{\sqrt{(p^2+1)(q^2+1)}} \end{aligned}$$

as we wanted.

An alternative approach is to use vectors and dot products to find $\cos\phi$. We have

$$\overrightarrow{FP}.\overrightarrow{FT} = FP.FT.\cos\phi$$

(where the dot on the left hand side is the dot product, but on the right is ordinary multiplication), so we need only find the lengths FP, FT as above and the dot product. The dot product is

$$\begin{aligned} \overrightarrow{FP}.\overrightarrow{FT} = \begin{pmatrix} ap^2-a \\ 2ap-0 \end{pmatrix} \cdot \begin{pmatrix} apq-a \\ a(p+q)-0 \end{pmatrix} &= (ap^2-a)(apq-a) + 2ap.a(p+q) \\ &= a^2(p^2-1)(pq-1) + 2a^2(p^2+pq) \\ &= a^2(p^3q - p^2 - pq + 1 + 2p^2 + 2pq) \\ &= a^2(p^3q + p^2 + pq + 1) \\ &= a^2(p^2(pq+1) + pq + 1) \\ &= a^2(p^2+1)(pq+1) \end{aligned}$$

Therefore we deduce

$$\begin{aligned} \cos\phi &= \frac{\overrightarrow{FP}.\overrightarrow{FT}}{FP.FT} \\ &= \frac{a^2(p^2+1)(pq+1)}{a(p^2+1).a\sqrt{(p^2+1)(q^2+1)}} \\ &= \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}} \end{aligned}$$

as required.

Marks

M1: Using the cosine rule or dot products to find $\cos\phi$
M1: Calculating TF^2 *or* $|TF|$
A1 cao: Fully factorising result correctly
M1: Calculating FP^2 *or* $|FP|$
A1 cao: Fully factorising result correctly
A1: Calculating TP^2 *or evaluating the dot product*
M1: Substituting in and simplifying to find $\cos\phi$ *(dependent on at least two of the preceding three method marks)*
A1 cso (AG): Determining $\cos\phi$ *as given in question*

[Total for finding $\cos\phi$: 8 marks]

Now to show that the line FT bisects the angle PFQ, it suffices to show that ϕ is equal to the angle TFQ (see the sketch above).

Now we can find $\cos(\angle TFQ)$ by using the above formula and swapping every p and q in it, as this will swap the roles of P and Q.

But swapping every p and q does not change the formula, so $\cos(\angle TFQ) = \cos(\angle TFP)$, and so $\angle TFQ = \angle TFP$ as both angles are strictly less than $180°$ and cosine is one-to-one in this domain.

Thus the line FT bisects the angle PFQ, as required.

Marks

M1: Swapping p and q to find $\cos(\angle TFQ)$ or calculating from scratch

A1: Showing $\cos(\angle TFQ) = \cos(\angle TFP)$

M1: Deducing that $\angle TFQ = \angle TFP$ (need some minimal comment or implication sign that cosines equal implies angles equal; does not need to state explicitly that $\phi < 180°$ for this mark or subsequent A1)

A1 cso: Making the required deduction about line TF

Alternative method: Calculating $\cos(\angle PFQ)$ first

M1: Calculating $\cos(\angle PFQ)$ using the cosine rule as above

A1 cao: Showing $\cos(\angle PFQ) = \dfrac{p^2q^2 - p^2 - q^2 + 4pq + 1}{(p^2+1)(q^2+1)}$

M1: Showing that $\cos(\angle PFQ) = \cos 2\phi$ using double-angle formula

A1 cso: Making the required deduction about line TF

[Total for this final part: 4 marks]

Question 5

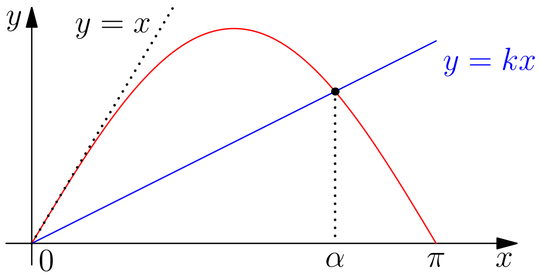
Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

We sketch the graph of $y = \sin x$ in the range $0 \leqslant x \leqslant \pi$ along with the line $y = kx$.

Now since $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, the gradient of $y = \sin x$ at $x = 0$ is 1, so the tangent at $x = 0$ is $y = x$. We therefore also sketch the line $y = x$.



It clear that there is at most one intersection of $y = kx$ with $y = \sin x$ in the interval $0 < x < \pi$, and since $0 < k < 1$, there is exactly one, as the gradient is positive and less that that of $y = \sin x$ at the origin. (If $k \leqslant 0$, there would be no intersections in this range as kx would be negative or zero; if $k \geqslant 1$, the only intersection would be at $x = 0$.)

Marks

M1: Determining gradient of $y = \sin x$ at $x = 0$

A1: Tangent at $x = 0$ is $y = x$ (award M1 A1 if just draw $y = x$ as tangent at origin, as long as it is labelled)

M1: Decent sketch showing $y = \sin x$ and $y = kx$ with $0 < k < 1$

A1: Reasonably convincing argument from sketch; must explain that $k < 1$ implies there is at least one intersection (condone ignoring the $0 < k$ condition) because gradient is less than that of $\sin x$ at origin

[Total for this part: 4 marks]

Let

$$I = \int_0^{\pi} \left|\sin x - kx\right| \mathrm{d}x\,.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

It is a pain to work with absolute values (the "modulus function"), so we split the integral into two integrals: in the interval $0 \leqslant x \leqslant \alpha$, $\sin x - kx \geqslant 0$, and in the interval $\alpha \leqslant x \leqslant \pi$,

$\sin x - kx \leqslant 0$. So

$$
\begin{aligned}
I &= \int_0^\pi \left|\sin x - kx\right| \mathrm{d}x \\
&= \int_0^\alpha \left|\sin x - kx\right| \mathrm{d}x + \int_\alpha^\pi \left|\sin x - kx\right| \mathrm{d}x \\
&= \int_0^\alpha \sin x - kx \,\mathrm{d}x + \int_\alpha^\pi -\sin x + kx \,\mathrm{d}x \\
&= \left[-\cos x - \tfrac{1}{2}kx^2\right]_0^\alpha + \left[\cos x + \tfrac{1}{2}kx^2\right]_\alpha^\pi \\
&= (-\cos\alpha - \tfrac{1}{2}k\alpha^2) - (-\cos 0 - 0) + (\cos\pi + \tfrac{1}{2}k\pi^2) - (\cos\alpha + \tfrac{1}{2}k\alpha^2) \\
&= -2\cos\alpha - k\alpha^2 + \tfrac{1}{2}k\pi^2 \\
&= -2\cos\alpha - \alpha\sin\alpha + \frac{\pi^2\sin\alpha}{2\alpha}
\end{aligned}
$$

where the last line follows using $k\alpha = \sin\alpha$ so that $k = (\sin\alpha)/\alpha$, and we have reached the desired result.

Marks

M1: Splitting integral into two appropriate intervals

M1 dep: Removing absolute values correctly

A1 cao: Correct integration of expression (ignoring limits; no follow through marks here)

M1: Substituting in to evaluate integral (condone one sign error)

M1: Correctly substituting $k = \sin\alpha/\alpha$ into expression

A1 cso (AG): Reaching given expression

[Total for this part: 6 marks]

Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

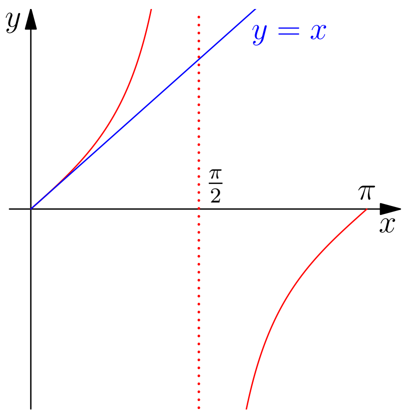
$$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

We differentiate I to find its stationary points. We have

$$
\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \frac{\pi^2}{2}\left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right) + 2\sin\alpha - \sin\alpha - \alpha\cos\alpha \\
&= (\sin\alpha - \alpha\cos\alpha) - \frac{\pi^2}{2\alpha^2}(\sin\alpha - \alpha\cos\alpha) \\
&= \left(1 - \frac{\pi^2}{2\alpha^2}\right)(\sin\alpha - \alpha\cos\alpha)
\end{aligned}
$$

so $\frac{\mathrm{d}I}{\mathrm{d}\alpha} = 0$ if and only if $2\alpha^2 = \pi^2$ or $\sin\alpha = \alpha\cos\alpha$. The former condition gives $\alpha = \pm\pi/\sqrt{2}$, while the latter condition gives $\tan\alpha = \alpha$.

A quick sketch of the tan graph (see below) shows that $\tan\alpha = \alpha$ has no solutions in the range $0 < \alpha < \pi$ (though $\alpha = 0$ is a solution); the sketch uses the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$, so the tangent to $y = \tan x$ at $x = 0$ is $y = x$.



Thus the only solution in the required range is $\alpha = \pi/\sqrt{2}$ (and note that $\pi/\sqrt{2} < \pi$).

> ***Marks***
> *Determining location of stationary point:*
>
> *M1: Finding $\mathrm{d}I/\mathrm{d}\alpha$ using quotient rule or equivalent*
> *M1: Factorising to solve $\mathrm{d}I/\mathrm{d}\alpha = 0$, performing at least one step and effectively dividing or completely factorising*
> *M1: Reasonable attempt to reject possibility $\sin\alpha = \alpha\cos\alpha$, e.g., by considering $\tan\alpha = \alpha$*
> *A1: Correctly rejecting this possibility*
> *A1 cao: Deducing $\alpha = \pi/\sqrt{2}$, dependent upon explicit rejection of $\sin\alpha = \alpha\cos\alpha$ possibility (can award even if previous M1 A1 is not awarded, as long as say that this case is not possible)*
> *[Total: 5 marks for finding stationary point]*

To ascertain whether it is a maximum, a minimum or a point of inflection, we could either look at the values of I or $\mathrm{d}I/\mathrm{d}\alpha$ at this point and either side or we could consider the second derivative.

Either way, we will eventually have to work out the value of I when $\alpha = \pi/\sqrt{2}$, so we will do so now:

$$\begin{aligned} I &= \frac{\pi^2 \sin(\pi/\sqrt{2})}{2\pi/\sqrt{2}} - 2\cos\frac{\pi}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} \\ &= \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} - 2\cos\frac{\pi}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} \\ &= -2\cos\frac{\pi}{\sqrt{2}}. \end{aligned}$$

Approach 1: Using values either side

The function I is not well-defined when $\alpha = 0$, but if we know that $\frac{\sin\alpha}{\alpha} \to 1$ as $\alpha \to 0$, we can deduce that as $\alpha \to 0$, $I \to -2 + \frac{\pi^2}{2} > \frac{5}{2} > 2$ (using $\pi > 3$).

When $\alpha = \pi$, we have $I = 2$.

Since at $\alpha = \pi/\sqrt{2}$, we have $I = -2\cos(\pi/\sqrt{2}) < 2$, this must be the minimum value of I.

Alternatively, if we wish to consider the value of $\mathrm{d}I/\mathrm{d}\alpha$, we need to know the sign of $\sin\alpha - \alpha\cos\alpha$ near $\alpha = \pi/\sqrt{2}$. Now since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$ and so this expression is positive. Therefore for α slightly less that $\pi/\sqrt{2}$, $\mathrm{d}I/\mathrm{d}\alpha < 0$ and for $\alpha > \pi/\sqrt{2}$, $\mathrm{d}I/\mathrm{d}\alpha > 0$, so that $\alpha = \pi/\sqrt{2}$ is a (local) minimum.

Approach 2: Using the second derivative

We have

$$\begin{aligned}\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} &= \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha) + \left(1 - \frac{\pi^2}{2\alpha^2}\right)(\cos\alpha - \cos\alpha + \alpha\sin\alpha) \\ &= \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha) + \left(1 - \frac{\pi^2}{2\alpha^2}\right)\alpha\sin\alpha\end{aligned}$$

Now when $\alpha = \pi/\sqrt{2}$, so that $\pi^2/2\alpha^2 = 1$, we have

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} = \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha).$$

Since $\alpha = \frac{\pi}{2}\sqrt{2} > \frac{\pi}{2}$, we have $\sin\alpha > 0$ and $\cos\alpha < 0$, so $\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} > 0$ and I has a local minimum at this value of α.

Marks

Evaluating the minimum:

B1 cso (AG): reaching the given expression for minimum I; can be awarded at any point.

Determining nature of stationary point:

Approach 1a: Evaluating I either side

[B1 for evaluating I at the stationary point is awarded below]
M1: Attempting to evaluate I at points either side of $\alpha = \pi/\sqrt{2}$
A1: Correctly evaluating on one side
A1: Correctly evaluating on other side
A1 cso: Convincingly deducing that $\alpha = \pi/\sqrt{2}$ gives a minimum

Approach 1b: Evaluating $\mathrm{d}I/\mathrm{d}\alpha$ either side
M1: Attempting to determine the values or signs of $\mathrm{d}I/\mathrm{d}\alpha$ at points either side of $\alpha = \pi/\sqrt{2}$
A1: Determining the sign of $\sin\alpha - \alpha\cos\alpha$ near $\pi/\sqrt{2}$
A1: Determining the sign of $1 - \pi^2/2\alpha^2$ either side of $\pi/\sqrt{2}$
A1 cso: Convincingly showing that $\alpha = \pi/\sqrt{2}$ gives a minimum

Approach 2: Second derivative

M1: Differentiating their expression for first derivative using product rule or otherwise
M1: Simplifying the evaluated expression in the case $\alpha = \pi/\sqrt{2}$
M1: Considering signs of $\cos\alpha$, $\sin\alpha$ for $\alpha = \pi/\sqrt{2}$
A1 cso: Convincingly deducing that the second derivative is positive
[Total for this part: 5 marks]

Question 6

Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

Using the formula in the formula book for the binomial expansion, we find that the x^r term is

$$\begin{aligned}\binom{-3}{r}(-x)^r &= \frac{(-3)(-4)(-5)\dots(-3-r+1)}{r!}(-1)^r x^r \\ &= \frac{3.4.5.\cdots.(r+2)}{r!}x^r \\ &= \frac{3.4.5.\cdots.(r+2)}{1.2.3.4.5.\cdots.r}x^r \\ &= \frac{(r+1)(r+2)}{1.2}x^r\end{aligned}$$

so the coefficient of x^r is $\frac{1}{2}(r+1)(r+2)$. But the argument as we've written it assumes that $r \geqslant 2$ (as we've left ourselves with "1.2" in the denominator), so we need to check that this this also holds for $r=0$ and $r=1$. But this is easy, as $\binom{-3}{0}(-1)^0 = 1 = \frac{1}{2} \times 1 \times 2$ and $\binom{-3}{1}(-1)^1 = 3 = \frac{1}{2} \times 2 \times 3$.

Alternatively, we could have argued

$$\frac{3.4.5.\cdots.(r+2)}{r!}x^r = \frac{1.2.3.4.5.\cdots.(r+2)}{1.2.r!}x^r = \frac{(r+1)(r+2)}{1.2}x^r$$

and this would have dealt with the cases $r=0$ and $r=1$ automatically, as we are not implicitly assuming that $r \geqslant 2$.

Marks

M1: Use of binomial theorem in this case, with the right structure of the expanded general coefficient (but condone, for example, an incorrect final term in the numerator)

A1 cso: Fully correct unsimplified coefficient

M1: Explicitly cancelling the negatives correctly

A1 cso (AG): Reaching the given expression for coefficient

A1 (dep on previous A1): Justifying that the given expression also holds for $r=0$ and $r=1$ (award together with previous A1 if the assumption $r \geqslant 2$ is not implicitly used)

[Total for this part: 5 marks]

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \cdots.$$

We have

$$\frac{1-x+2x^2}{(1-x)^3} = (1-x+2x^2)(a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \dots)$$

where $a_r = \frac{1}{2}(r+1)(r+2)$. Thus

$$\begin{aligned}\frac{1-x+2x^2}{(1-x)^3} &= a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots \\ &\qquad - a_0x - a_1x^2 - \cdots - a_{r-1}x^r - \cdots \\ &\qquad\qquad + 2a_0x^2 + \cdots + 2a_{r-2}x^r + \cdots \\ &= a_0 + (a_1-a_0)x + (a_2-a_1+2a_0)x^2 + \cdots \\ &\qquad + (a_r - a_{r-1} + 2a_{r-2})x^r + \cdots\end{aligned}$$

Thus the coefficient of x^r for $r \geqslant 2$ is

$$\begin{aligned}a_r - a_{r-1} + 2a_{r-2} &= \tfrac{1}{2}(r+1)(r+2) - \tfrac{1}{2}r(r+1) + (r-1)r \\ &= \tfrac{1}{2}(r^2+3r+2-r^2-r+2r^2-2r) \\ &= \tfrac{1}{2}(2r^2+2) \\ &= r^2+1\end{aligned}$$

as required. Also, the coefficient of x^0 is $a_0 = 1 = 0^2+1$ and the coefficient of x^1 is $a_1 - a_0 = 3-1 = 2 = 1^2+1$, so the formula r^2+1 holds for these two cases as well. Therefore, the coefficient of x^r is r^2+1 for all $r \geqslant 0$.

Now we can sum our series: it is

$$\begin{aligned}1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \cdots &= \frac{0^2+1}{2^0} + \frac{1^2+1}{2^1} + \frac{2^2+1}{2^2} + \cdots + \frac{r^2+1}{2^r} + \cdots \\ &= (0^2+1) + (1^2+1)(\tfrac{1}{2}) + \cdots + (r^2+1)(\tfrac{1}{2})^r + \cdots \\ &= \frac{1-\frac{1}{2}+2(\frac{1}{2})^2}{(1-\frac{1}{2})^3} \\ &= \frac{1}{(\frac{1}{8})} \\ &= 8.\end{aligned}$$

Marks

M1: Writing the quotient as a product $(1-x+2x^2)(a_0+\cdots)$, possibly with numerical or algebraic coefficients in the latter bracket; must include a general term, though

M1 dep: Expanding the product, in particular showing the coefficients of x^0, x^1 and x^r

A1: Correct coefficient of x^r in expansion (in unsimplified form)

A1 cso (AG): Deducing the required expression for $r \geqslant 2$ (with or without specifying this condition)

B1: Checking that general expression holds for $r=0$ and $r=1$

[Total for verifying r^2+1: 5 marks]

B1: For general term $(r^2+1)/2^r$

M1: Substituting $x = \frac{1}{2}$ into formula

A1 cao: Sum is 8

[Total for summing series: 3 marks]

> *(ii)* *Find the sum of the series*
> $$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}+\cdots.$$

The denominators look like powers of 2, so we will rewrite the terms using powers of 2:

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\cdots=\frac{1}{1}+\frac{4}{2}+\frac{9}{4}+\frac{16}{8}+\frac{25}{16}+\frac{36}{32}+\frac{49}{64}+\cdots$$

and it is clear that the general term is $r^2/2^{r-1}$, starting with the term where $r=1$.

We can rewrite this in terms of the series found in part (i) by writing

$$\frac{r^2}{2^{r-1}}=2\cdot\frac{r^2}{2^r}=2\cdot\frac{r^2+1}{2^r}-2\cdot\frac{1}{2^r},$$

so our series becomes

$$\begin{aligned}
2&\left(\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\cdots\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)\\
&=2\left(1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\cdots\right)-2\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)\\
&=2\cdot 8-2\cdot 2\\
&=12,
\end{aligned}$$

where on the second line, we have introduced the term corresponding to $r=0$, and on the penultimate line, we have used the result from (i) and the sum of the infinite geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1/(1-\frac{1}{2})=2$.

An alternative approach is to begin with the result of part (i) and to argue as follows.

We have

$$\begin{aligned}
\frac{1-x+2x^2}{(1-x)^3}&=\sum_{r=0}^{\infty}(r^2+1)x^r\\
&=\sum_{r=0}^{\infty}r^2x^r+\sum_{r=0}^{\infty}x^r\\
&=x\sum_{r=0}^{\infty}r^2x^{r-1}+\sum_{r=0}^{\infty}x^r.
\end{aligned}$$

But our required sum is $\sum_{r=0}^{\infty}r^2(\frac{1}{2})^{r-1}$, so we put $x=\frac{1}{2}$ into this result and get

$$\frac{1-\frac{1}{2}+2(\frac{1}{2})^2}{(1-\frac{1}{2})^3}=\frac{1}{2}\sum_{r=0}^{\infty}r^2(\tfrac{1}{2})^{r-1}+\sum_{r=0}^{\infty}(\tfrac{1}{2})^r.$$

The last term on the right hand side is our geometric series, summing to 2. The left hand side evaluates to 8, and so we get

$$8=\frac{1}{2}\sum_{r=0}^{\infty}r^2(\tfrac{1}{2})^{r-1}+2.$$

Thus our series sums to 12, as before.

A third approach is to observe that the series can be written as $\sum_{r=0}^{\infty}(r+1)^2 x^r = \sum_{r=0}^{\infty}(r^2 + 2r + 1)x^r$ with $x = \frac{1}{2}$, then to look for a polynomial $\mathrm{p}(x)$ of degree at most 2 such that the coefficient of x^r in the expansion of $\mathrm{p}(x)/(1-x)^3$ is exactly $r^2 + 2r + 1$, using methods like those in part (i). (The polynomial needs to be of degree at most 2 so that the terms are also correct for $r = 0$ and $r = 1$ in addition to the general term being correct.) This turns out to give $\mathrm{p}(x) = x + 1$, so that the sum is $(\frac{1}{2}+1)/(1-\frac{1}{2})^3 = 12$.

Marks

Markscheme for Approach 1:

M1: Identifying the general term as $r^2/2^{r-1}$ or equivalent

M1 dep: Splitting $r^2/2^{r-1}$ into two terms to match part (i)

M1 dep: Identifying the "missing" first term

M1: Sum infinite GP for second bracket

A1 cao: Sum of infinite GP

M1: Using the result of part (i) to sum the first bracket

A1: Reaching the answer 12; can follow through an incorrect answer to (i) if it is used in the solution to this part

[Total for part (ii): 7 marks]

Question 7

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ *when* $|x|$ *is small.*

The height of the water in a tank at time t *is* h*. The initial height of the water is* H *and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.*

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$*, where* α *is a constant greater than 1, the height remains constant. Show that*

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h),$$

for some positive constant k*. Deduce that the time* T *taken for the water to reach height* αH *is given by*

$$kT = \ln\left(1 + \frac{1}{\alpha}\right)$$

and that $kT \approx \alpha^{-1}$ *for large values of* α*.*

Since the tank has constant cross-sectional area, the volume of water within the tank is proportional to the height of the water.

Therefore we have the height increasing at a rate $a - bh$, where a is the rate of water flowing in divided by the cross-sectional area, and b is a constant of proportionality representing the rate of water leaking out. In other words, we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - bh.$$

Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\alpha^2 H = 0$, or $a = b\alpha^2 H$, giving

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha^2 H - bh = b(\alpha^2 H - h).$$

Hence if we write $k = b$, we have our desired equation.

We can now solve this by separating variables to get

$$\int \frac{1}{\alpha^2 H - h}\,\mathrm{d}h = \int k\,\mathrm{d}t$$

so that

$$-\ln(\alpha^2 H - h) = kt + c.$$

At $t = 0$, $h = H$, so

$$-\ln(\alpha^2 H - H) = c,$$

which finally gives us

$$kt = \ln(\alpha^2 H - H) - \ln(\alpha^2 H - h).$$

Now at time T, $h = \alpha H$, so that

$$\begin{aligned} kT &= \ln(\alpha^2 H - H) - \ln(\alpha^2 H - \alpha H) \\ &= \ln\left(\frac{\alpha^2 H - H}{\alpha^2 H - \alpha H}\right) \\ &= \ln\left(\frac{\alpha^2 - 1}{\alpha^2 - \alpha}\right) \\ &= \ln\left(\frac{\alpha + 1}{\alpha}\right) \\ &= \ln\left(1 + \frac{1}{\alpha}\right) \end{aligned}$$

as required.

When α is large, so that $\frac{1}{\alpha}$ is small, this is

$$\begin{aligned} kT &= \ln\left(1 + \frac{1}{\alpha}\right) \\ &\approx \frac{1}{\alpha} - \frac{1}{2\alpha^2} \\ &\approx \frac{1}{\alpha}. \end{aligned}$$

Marks

M1: Differential equation in form $\mathrm{d}h/\mathrm{d}t = a - bh$ or equivalent
M1 dep: Deducing $a - b\alpha^2 H = 0$ to find a
A1 cso (AG): Reaching stated differential equation (ODE)
M1: Separating variables
A1 cao: Deducing general solution to ODE
M1: Deducing specific solution to ODE using boundary conditions
M1: Finding expression for kT (in form $\ln(\cdots) - \ln(\cdots)$ or better)
A1 cso: Reaching $kT = \ln(1 + \frac{1}{\alpha})$ as given
B1 cso (AG): Using given approximation for $\ln(1 + x)$ to reach stated approximation $kT \approx \alpha^{-1}$

Alternative for marks 4–6 (M1 A1 M1) if use $\int_0^T \ldots \mathrm{d}t = \int_H^{\alpha H} \ldots \mathrm{d}h$:
M1: Separating variables
M1: Limits essentially correct
A1: Integration to reach $-\ln(\ldots)$
[Total for part (i): 9 marks]

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c and that $cT' \approx \sqrt{H}$ for large values of α.

We proceed just as in part (i).

This time we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - b\sqrt{h},$$

where a and b are some constants. Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\sqrt{\alpha^2 H} = 0$, which yields $a = b\alpha\sqrt{H}$. We thus have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha\sqrt{H} - b\sqrt{h} = b(\alpha\sqrt{H} - \sqrt{h}).$$

So if this time we write $c = b$, we have our desired differential equation.

We again solve this by separating variables to get

$$\int \frac{1}{\alpha\sqrt{H} - \sqrt{h}}\,\mathrm{d}h = \int c\,\mathrm{d}t.$$

To integrate the left hand side, we use the substitution $u = \sqrt{h}$, so that $h = u^2$ and $\frac{\mathrm{d}h}{\mathrm{d}u} = 2u$. This gives us

$$\int \frac{1}{\alpha\sqrt{H} - u} \cdot 2u\,\mathrm{d}u = ct.$$

We divide the numerator by the denominator to get

$$\begin{aligned}
ct &= \int \frac{-2(\alpha\sqrt{H} - u) + 2\alpha\sqrt{H}}{\alpha\sqrt{H} - u}\,\mathrm{d}u \\
&= \int -2 + \frac{2\alpha\sqrt{H}}{\alpha\sqrt{H} - u}\,\mathrm{d}u \\
&= -2u - 2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H} - u\bigr) + c' \\
&= -2\sqrt{h} - 2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H} - \sqrt{h}\bigr) + c'
\end{aligned}$$

where c' is a constant.

An alternative way of doing this step is to use the substitution $v = \alpha\sqrt{H} - \sqrt{h}$, so that $h = (\alpha\sqrt{H} - v)^2 = \alpha^2 H - 2\alpha v\sqrt{H} + v^2$ and $\frac{\mathrm{d}h/\mathrm{d}v}{=} - 2\alpha\sqrt{H} + 2v$. This gives us

$$\begin{aligned}
ct &= \int \frac{1}{v}(-2\alpha\sqrt{H} + 2v) = ct \\
&= \int \frac{-2\alpha\sqrt{H}}{v} + 2\,\mathrm{d}v \\
&= -2\alpha\sqrt{H}\ln v + 2v + c' \\
&= -2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H} - \sqrt{h}\bigr) + 2\bigl(\alpha\sqrt{H} - \sqrt{h}\bigr) + c'
\end{aligned}$$

where c' is again a constant.

At $t = 0$, $h = H$, so

$$c' = 2\sqrt{H} + 2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H} - \sqrt{H}\bigr).$$

Now at time T', $h = \alpha H$, so that

$$\begin{aligned}
cT' &= -2\sqrt{\alpha H} - 2\alpha\sqrt{H}\ln\big(\alpha\sqrt{H} - \sqrt{\alpha H}\big) + 2\sqrt{H} + 2\alpha\sqrt{H}\ln\big(\alpha\sqrt{H} - \sqrt{H}\big) \\
&= 2\sqrt{H}(1-\sqrt{\alpha}) + 2\alpha\sqrt{H}\ln\left(\frac{\alpha\sqrt{H}-\sqrt{H}}{\alpha\sqrt{H}-\sqrt{\alpha H}}\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(\frac{(\sqrt{\alpha}+1)(\sqrt{\alpha}-1)}{\sqrt{\alpha}(\sqrt{\alpha}-1)}\right)\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}}\right)\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(1+\frac{1}{\sqrt{\alpha}}\right)\right)
\end{aligned}$$

as required.

When α is large, $1/\sqrt{\alpha}$ is small, so this gives

$$\begin{aligned}
cT' &= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(1+\frac{1}{\sqrt{\alpha}}\right)\right) \\
&\approx 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\left(\frac{1}{\sqrt{\alpha}}-\frac{1}{2\alpha}\right)\right) \\
&\approx 2\sqrt{H}\big(1-\sqrt{\alpha}+\sqrt{\alpha}-\tfrac{1}{2}\big) \\
&\approx \sqrt{H}.
\end{aligned}$$

Marks

M1: Correct structure of ODE ($\mathrm{d}h/\mathrm{d}t = a - b\sqrt{h}$)
A1: Correct ODE: $\mathrm{d}h/\mathrm{d}t = c(\alpha\sqrt{H} - \sqrt{h})$
M1: Reasonable attempt to substitute $u = \sqrt{h}$ *or similarly effective substitution*
M1 dep: Fully correct substitution
M1 dep: Dividing through numerator by denominator to simplify fraction
A1: Correct general solution to ODE, condone at most one algebraic error
A1 cao: Correct particular solution to ODE
M1: Substituting $h = \alpha H$ *correctly and simplifying logs to find* cT'
A1 cso (AG): Determining cT' *in form given in question*
M1: Applying approximation for $\ln(1+x)$ *correctly*
A1 cso (AG): Deducing approximation given in question

Alternative for marks 4–8 (M1 M1 A1 A1 M1) if use $\int_0^T \dots \mathrm{d}t = \int_H^{\alpha H} \dots \mathrm{d}h$:
M1 dep: Fully correct substitution, ignoring limits
M1 dep: Dividing through numerator by denominator to simplify fraction
M1: Changing limits, substantially correctly
A1: Correct integration, condone at most one algebraic error
A1 cao: Correct evaluation of integral

SC: If begin with $\mathrm{d}h/\mathrm{d}t = c(\alpha^2 H - \sqrt{h})$, *then can score max M0 A0 M1 M1 M1 A1 A0 M1 A0 M1 A1*

[Total for part (ii): 11 marks]

Question 8

> *(i)* *The numbers m and n satisfy*
> $$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$
>
> *(a) Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.*

As $n^2 \geqslant 0$, we have

$$\begin{aligned} m^3 &= n^3 + n^2 + 1 \\ &\geqslant n^3 + 1 \\ &> n^3 \end{aligned}$$

so $m > n$ as the function $\mathrm{f}(x) = x^3$ is strictly increasing.

Now

$$\begin{aligned} m < n+1 &\iff m^3 < (n+1)^3 \\ &\iff n^3 + n^2 + 1 < n^3 + 3n^2 + 3n + 1 \\ &\iff 0 < 2n^2 + 3n \end{aligned}$$

so $m < n+1$ if and only if $2n^2 + 3n > 0$.

Combining these two conditions, $n < m$ always, and $m < n+1$ if and only if $2n^2 + 3n > 0$, so $n < m < n+1$ unless $2n^2 + 3n \leqslant 0$.

Now $2n^2 + 3n = 2n(n + \frac{3}{2}) \leqslant 0$ if and only if $-\frac{3}{2} \leqslant n \leqslant 0$, so $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

Marks

M1: Using $n^2 \geqslant 0$ to show $m^3 > n^3$

A1 cso (AG): Correct deduction of $m > n$; be generous, but showing the converse ($m > n$ implies $m^3 > n^3 + n^2 + 1$) alone gets A0

M1: Cubing $m < n+1$ and expanding $(n+1)^3$

A1 cso (AG): Reaching given condition $2n^2 + 3n > 0$; must have an explicit "if and only if" argument for this mark

M1 dep on at least one previous M: Combining conditions to get $n < m < n+1$ unless $2n^2 + 3n \leqslant 0$

M1: Some reasonable method of solving this quadratic inequality

A1 cso (AG; dependent on final two M marks): Reaching correct conclusion ($-\frac{3}{2} \leqslant n \leqslant 0$) through a valid method

[Total for part (i)(a): 7 marks]

> *(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.*

If solution to $(*)$ has both m and n integer, we cannot have $n < m < n+1$, as there is no integer strictly between two consecutive integers. We therefore require $-\frac{3}{2} \leqslant n \leqslant 0$, so $n = -1$ or $n = 0$.

If $n = -1$, then $m^3 = 1$, so $m = 1$.

If $n = 0$, then $m^3 = 1$, so $m = 1$.

Thus the only integer solutions are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

Marks
M1: Deducing that $n = -1$ or $n = 0$
M1 dep: Substituting into $(*)$ to determine m
A1 cso (AG): Reaching given answers
[Total for part (i)(b): 3 marks]

> *(ii)* *Find all integer solutions of the equation*
> $$p^3 = q^3 + 2q^2 - 1\,.$$

We try a similar argument here. We start by determining whether $p > q$:

$$\begin{aligned} p > q &\iff p^3 > q^3 \\ &\iff q^3 + 2q^2 - 1 > q^3 \\ &\iff 2q^2 > 1 \\ &\iff q^2 > \tfrac{1}{2} \end{aligned}$$

so that $p > q$ unless $q^2 \leqslant \frac{1}{2}$, and $q^2 \leqslant \frac{1}{2}$ if and only if $-\frac{1}{\sqrt{2}} \leqslant q \leqslant \frac{1}{\sqrt{2}}$.

We now determine the conditions under which $p < q + 1$:

$$\begin{aligned} p < q + 1 &\iff p^3 < (q+1)^3 \\ &\iff q^3 + 2q^2 - 1 < q^3 + 3q^2 + 3q + 1 \\ &\iff 0 < q^2 + 3q + 2 \end{aligned}$$

so $p < q + 1$ unless $q^2 + 3q + 2 \leqslant 0$. This condition becomes $(q+1)(q+2) \leqslant 0$, so $-2 \leqslant q \leqslant -1$.

Thus $q < p < q + 1$ unless $-\frac{1}{\sqrt{2}} \leqslant q \leqslant \frac{1}{\sqrt{2}}$ or $-2 \leqslant q \leqslant -1$.

If p and q are both integers, this then limits us to three cases: $q = 0$, $q = -1$ and $q = -2$.

If $q = 0$, then $p^3 = -1$, so $p = -1$.

If $q = -1$, then $p^3 = 0$, so $p = 0$.

If $q = -2$, then $p^3 = -1$, so $p = -1$.

Hence there are three integer solutions: $(p, q) = (-1, 0)$, $(p, q) = (-1, -2)$ and $(p, q) = (0, -1)$.

Marks
M1: Checking the conditions for $p > q$ by cubing this condition
A1: Reaching the conclusion $p > q$ implies $2q^2 > 1$ or similar; logic has to follow later on to get further marks
A1: Either deduce that $p > q$ unless $-\frac{1}{\sqrt{2}} \leqslant q \leqslant \frac{1}{\sqrt{2}}$ or that $p > q$ with p and q integers unless $q = 0$ ("condition 1")
M1: Checking the conditions for $p < q + 1$ by cubing this condition
A1: Reaching the conclusion $p < q + 1$ implies $(q+1)(q+2) > 0$ or similar argument; again logic must follow later for further credit

A1: Deducing $p < q + 1$ unless $-2 \leqslant q \leqslant -1$ ("condition 2")

A1 ft: Combining conditions to get $q < p < q + 1$ unless condition 1 OR condition 2 is met (follow through conditions, but do not condone use of AND in place of OR); requires correct logic for this mark

M1: Checking all of their *possible values of q to determine possible values of p*

A1: At least two correct solutions (dependent on getting the previous M1, so if they aren't checking all of their possible q values, they cannot get this mark!)

A1 cso: All three correct (p, q) pairs with correct working (condone lack of "iff" argument earlier for this final mark)

SC: All three solutions correctly given with no justification that these are all the integer solutions scores B1 only.

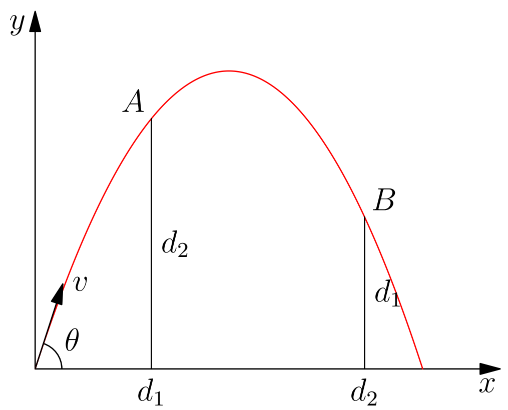
[Total for part (ii): 10 marks]

Question 9

A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}\,.$$

We draw a sketch of the situation:



We let the speed of projection be v and the time from launch be t. We resolve the components of velocity to find the position (x, y) at time t:

$$\begin{aligned} &\mathscr{R}(\rightarrow) \qquad & x &= (v\cos\theta)t && (1)\\ &\mathscr{R}(\uparrow) \qquad & y &= (v\sin\theta)t - \tfrac{1}{2}gt^2 && (2) \end{aligned}$$

At A (distance d_1 from the point of projection), we find

$$(v\cos\theta)t = d_1$$
$$(v\sin\theta)t - \tfrac{1}{2}gt^2 = d_2$$

so that

$$t = \frac{d_1}{v\cos\theta}$$

giving

$$\frac{v\sin\theta}{v\cos\theta}d_1 - \frac{\frac{1}{2}gd_1^2}{v^2\cos^2\theta} = d_2,$$

so that

$$d_2 = d_1\tan\theta - \frac{gd_1^2}{2v^2\cos^2\theta}.$$

This can be rearranged to get

$$\frac{gd_1^2}{2v^2\cos^2\theta} = d_1\tan\theta - d_2. \qquad (3)$$

(An alternative is to first eliminate t from equations (1) and (2) first to get

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} \qquad (4)$$

and then substitute $x = d_1$ and $y = d_2$ into this formula.)

Likewise, at B we get (on swapping d_1 and d_2):

$$\frac{gd_2^2}{2v^2\cos^2\theta} = d_2\tan\theta - d_1. \tag{5}$$

Multiplying (3) by d_2^2 gives the same left hand side as when we multiply (5) by d_1^2, so that

$$(d_1\tan\theta - d_2)d_2^2 = (d_2\tan\theta - d_1)d_1^2.$$

Expanding this gives

$$d_1d_2^2\tan\theta - d_2^3 = d_1^2d_2\tan\theta - d_1^3.$$

Collecting terms gives:

$$(d_1d_2^2 - d_1^2d_2)\tan\theta = d_2^3 - d_1^3,$$

and we can factorise this (recalling that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$) to get

$$d_1d_2(d_2-d_1)\tan\theta = (d_2-d_1)(d_2^2 + d_2d_1 + d_1^2).$$

Dividing by $d_1d_2(d_2-d_1) \neq 0$ gives us our desired result:

$$\tan\theta = \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2}.$$

Marks

M1: Drawing a sketch to show the setup, and introducing the initial speed v; if no sketch, award this method mark if next answer marks are awarded

A1 cao: Writing down horizontal and vertical components of displacement (can be awarded if the general expression is not seen, but the results are used in the next step)

M1: Evaluating these at A (or B) to get equations for d_1 and d_2 in terms of t, v, g and θ

M1: Finding the time at which the particle reaches A (or B) in terms of d_1 (or d_2), θ and v (this can be awarded if the result is implicitly used to eliminate t)

A1 cao: Eliminating t from the equations to get $d_2 = d_1\tan\theta - \cdots$ or equivalent

An alternative approach to the previous three marks is:

M1: Rearranging the x formula to get $t = \cdots$ (can be awarded if the result is used implicitly in the next step)

M1 dep: Substituting $t = \cdots$ into the formula for y to get a formula for y in terms of x

A1 cao: Substituting $x = d_1$, $y = d_2$ (or vice versa) to get $d_2 = d_1\tan\theta - \cdots$ or equivalent

B1 ft: Writing down the corresponding equation for the second wall

M1: Eliminating v from the pair of equations to get an equation involving d_1, d_2 and θ only

A1 cao: Deducing a correct equation involving d_1, d_2 and $\tan\theta$ only from their earlier equations

M1: Rearranging to find $\tan\theta$ in terms of d_1 and d_2

M1 dep: Factorising resulting expression or equation to simplify it

A1 cso (AG): Reaching given equation

[Total for this part: 11 marks]

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

The range can be found by determining where $y = 0$, so $(v \sin\theta)t - \frac{1}{2}gt^2 = 0$. This has solutions $t = 0$ (the point of projection) and $t = (2v/g)\sin\theta$. At this point,

$$\begin{aligned} x = (v\cos\theta)t &= \frac{2v^2 \sin\theta\cos\theta}{g} \\ &= \frac{2v^2\cos^2\theta}{g}\tan\theta. \end{aligned}$$

We have written $\sin\theta = \cos\theta\tan\theta$ because equation (3) gives us a formula for the fraction part of this expression: we get

$$x = \frac{d_1^2}{d_1\tan\theta - d_2}\tan\theta.$$

Alternatively, using equation (4), we can solve for $y = 0$ to get $x = 0$ or

$$\tan\theta = \frac{gx}{2v^2\cos^2\theta}.$$

Since $x = 0$ at the start, the other solution gives the range. Using equation (3) to write

$$\frac{g}{2v^2\cos^2\theta} = \frac{d_1\tan\theta - d_2}{d_1^2},$$

we deduce that

$$x = \frac{d_1^2\tan\theta}{d_1\tan\theta - d_2}$$

as before.

We can now simply substitute in our formula for $\tan\theta$, simplify a little, and we will be done:

$$\begin{aligned} x &= \frac{d_1^2\left(\dfrac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2}\right)}{d_1\left(\dfrac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2}\right) - d_2} \\ &= \frac{d_1(d_1^2 + d_1d_2 + d_2^2)}{(d_1^2 + d_1d_2 + d_2^2) - d_2^2} \\ &= \frac{d_1(d_1^2 + d_1d_2 + d_2^2)}{d_1^2 + d_1d_2} \\ &= \frac{d_1^2 + d_1d_2 + d_2^2}{d_1 + d_2} \end{aligned}$$

Alternative approach

An entirely different approach to the whole question is as follows. We know that the path of the projectile is a parabola. Taking axes as in the above sketch, the path passes through the three points $(0, 0)$, (d_1, d_2) and (d_2, d_1). If the equation of the curve is $y = ax^2 + bx + c$, then this gives three simultaneous equations:

$$\begin{aligned} 0 &= 0a + 0b + c \\ d_2 &= d_1^2 a + d_1 b + c \\ d_1 &= d_2^2 a + d_2 b + c. \end{aligned}$$

The first gives $c = 0$, and we can then solve the other two equations to get a and b. This gives

$$\begin{aligned} a &= \frac{d_2^2 - d_1^2}{d_1^2 d_2 - d_2^2 d_1} = -\frac{d_1 + d_2}{d_1 d_2} \\ b &= \frac{d_1^3 - d_2^3}{d_1^2 d_2 - d_2^2 d_1} = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2} \end{aligned}$$

Then the gradient is given by $\mathrm{d}y/\mathrm{d}x = 2ax + b$, so at $x = 0$, the gradient $\mathrm{d}y/\mathrm{d}x = b$, which gives us $\tan\theta$ (as the gradient is the tangent of angle made with the x-axis). The range is given by solving $y = 0$, so $x(ax + b) = 0$, giving $x = -b/a = (d_1^2 + d_1 d_2 + d_2^2)/(d_1 + d_2)$ as before.

Marks

M1: Solving $y = 0$ to find range (either in equation for y in terms of x or in terms of t)
A1: Finding t at $y = 0$ in terms of v, θ and g
M1: Finding range by substituting t into x

Alternative for preceding 2 marks if $y = x\tan\theta - \dots$ is used:
M1: Using $x \neq 0$ to find correct solution
A1: Deducing $gx/2v^2\cos^2\theta = \tan\theta$

M1: Using (3) to eliminate v and g
A1 ft: Correct expression for range involving d_1, d_2 and $\tan\theta$ only
M1: Substituting for $\tan\theta$ to leave a formula in terms of d_1 and d_2 only
A1 cao: Correct (unsimplified) formula for range in terms of d_1 and d_2 only
M1: Simplifying resulting fraction
A1 cao: Simplified expression for range
[Total for this part: 9 marks]

Question 10

A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic. Explain why the speeds of A and B immediately before the first collision are the same.

Assume they collide at height $H < h$. The perfectly elastic bounce means that there was no loss of energy, so A has the same total energy at height H on its upwards journey as it did when travelling downwards. We can work out the speeds at the point of collision, calling them v_A and v_B for A and B respectively. We write M for the mass of A and m for the mass of B (as in the next part of the question). We have, by conservation of energy

$$\begin{aligned} MgH + \tfrac{1}{2}Mv_A^2 &= Mgh \\ mgH + \tfrac{1}{2}mv_B^2 &= mgh \end{aligned}$$

so that $v_A^2 = 2(gh - gH)$ and $v_B^2 = 2(gh - gH)$, so $|v_A| = |v_B|$ and the speeds of A and B are the same.

Marks

B1: No loss of energy at bounce

M1: Use of conservation of energy to find v_A or similar

M1: Determining v_A or v_A^2 (ignore the sign of v_A if square roots are taken)

A1 (dependent on M marks only; can award even if B0): Correct deduction of result; condone $v_A = v_B$.

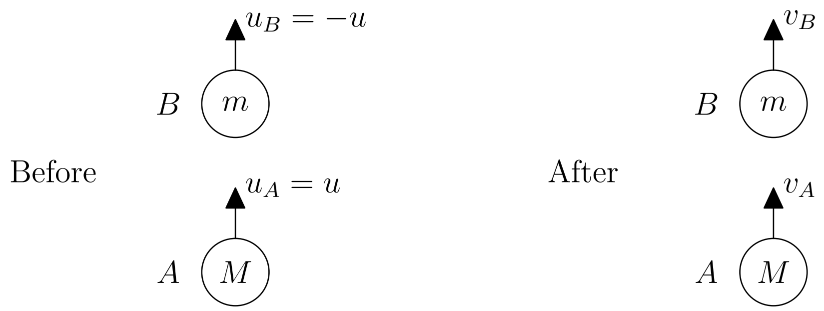
[Total for this explanation: 4 marks]

The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M-m)u^2}{(M+m)^2 g}\,.$$

*This begins as a standard collision of particles question, and so I will repeat the advice from the 2010 mark scheme: **ALWAYS** draw a diagram for collisions questions; you will do yourself (and the marker) no favours if you try to keep all of the directions in your head, and you are very likely to make a mistake. My recommendation is to always have all of the velocity arrows pointing in the same direction. In this way, there is no possibility of getting the signs wrong in the Law of Restitution: it **always** reads $v_1 - v_2 = e(u_2 - u_1)$ or $\dfrac{v_1 - v_2}{u_2 - u_1} = e$, and you only have to be careful with the signs of the given velocities. The algebra will then keep track of the directions of the unknown velocities for you.*

A diagram showing the first collision is as follows.



Then Conservation of Momentum gives

$$Mu_A + mu_B = Mv_A + mv_B$$

so

$$Mu - mu = Mv_A + mv_B,$$

and Newton's Law of Restitution gives

$$v_B - v_A = 1(u_A - u_B)$$

(using $e = 1$ as the collision is perfectly elastic). Substituting $u_A = u$ and $u_B = -u$ gives

$$Mv_A + mv_B = (M - m)u \tag{1}$$
$$v_B - v_A = 2u. \tag{2}$$

Then solving these equations (by (1) $- m \times$ (2) and (1) $+ M \times$ (2)) gives

$$v_A = \frac{(M - 3m)u}{M + m} \tag{3}$$
$$v_B = \frac{(3M - m)u}{M + m}. \tag{4}$$

To show that both particles move upwards after their first collision, we need to show that $v_A > 0$ and $v_B > 0$. From equation (3) and $M > 3m$ (given in the question), we see that $v_A > 0$; from equation (4) and $3M - m > 9m - m > 0$ (as $M > 3m$), we see that $v_B > 0$. Thus both particles move upwards after their first collision.

Marks

B1: Correct Conservation of Momentum (CoM) equation

M1: Using Law of Restitution (LoR) correctly in context (i.e., with $e = 1$; condone at most one sign error for this mark)

A1 cao: Correct resulting equation with u substituted appropriately; the signs must be consistent between the CoM and LoR equations for this mark

M1: Solving equations simultaneously

A1 cao: Correct v_A

A1 cao: Correct v_B

M1: Condition on at least one of v_A and v_B for particles to be going upwards

A1 cso: Justification that $v_A > 0$

A1 cso: Justification that $v_B > 0$ (needs more than just stating $3M - m > 0$)

[Total for showing both particles move upwards: 9 marks]

To find the maximum height of B between the two collisions, we begin by finding the maximum height that would be achieved by B following the first collision assuming that there is no second collision. We then explain why the second collision occurs during B's subsequent downward motion and deduce that it reaches that maximum height between the collisions.

The kinetic energy (KE) of B before the first collision is $\frac{1}{2}mu^2$ and after the first collision is

$$\tfrac{1}{2}mv_B^2 = \tfrac{1}{2}m\left(\frac{3M-m}{M+m}\right)^2 u^2,$$

so that B has a gain in KE of $\frac{1}{2}mv_B^2 - \frac{1}{2}mu^2$. When B is again at height h above the plane, which is where it was dropped from, it now has this gain as its KE. (This is because the KE just before the first collision has come from the loss of GPE; when the particle is once again at height h, this original KE ($\frac{1}{2}mu^2$) has been converted back into GPE.)

The particle B can therefore rise by a further height of H, where

$$mgH = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mu^2 = \tfrac{1}{2}mu^2\left(\left(\frac{3M-m}{M+m}\right)^2 - 1\right),$$

so

$$\begin{aligned} H &= \frac{u^2}{2g}\left(\frac{9M^2-6Mm+m^2}{(M+m)^2} - \frac{M^2+2Mm+m^2}{(M+m)^2}\right) \\ &= \frac{u^2}{2g}\left(\frac{8M^2-8Mm}{(M+m)^2}\right) \\ &= \frac{4u^2}{g}\left(\frac{M(M-m)}{(M+m)^2}\right). \end{aligned}$$

Thus the maximum height reached by B after the first collision, assuming that the second collision occurs after B has started falling is

$$h + H = h + \frac{4M(M-m)u^2}{(M+m)^2 g}.$$

Finally, we have to explain why A does not catch up with B before B begins to fall. But this is easy: B initially has a greater upward velocity than A (as $v_B - v_A = 2u > 0$), so the height of B is always greater than the height of A. Therefore they can only collide again after A has bounced on the ground and is in its ascent while B is in its descent.

Alternative approach: using constant acceleration and "suvat"

An alternative approach is to use the formulæ for constant acceleration ("suvat"), as follows.

Just before collision, B has speed u, so the height H of B at this point is given by the "suvat" equation $v^2 = u^2 + 2as$, taking positive to be downwards:

$$u^2 = 0^2 + 2g(h-H),$$

giving $H = h - u^2/2g$.

Immediately after the collision, B has velocity upwards given by equation (4) above. At the maximum height, $h_{\max}$, the speed of B is zero, so we can determine the maximum height using $v^2 = u^2 + 2as$ again; this time, we take positive to be upwards, so $a = -g$:

$$0^2 = \left(\frac{(3M-m)u}{M+m}\right)^2 - 2g(h_{\max} - H).$$

(Note that $h_{\max} - H > 0$.)

Rearranging this gives

$$\begin{aligned} h_{\max} &= H + \frac{1}{2g}\left(\frac{(3M-m)u}{M+m}\right)^2 \\ &= h - \frac{u^2}{2g} + \frac{u^2}{2g}\left(\frac{(3M-m)}{M+m}\right)^2 \\ &= h + \frac{u^2}{2g}\left(\frac{(3M-m)^2-(M+m)^2}{(M+m)^2}\right) \\ &= h + \frac{u^2}{2g}\left(\frac{8M^2-8Mm}{(M+m)^2}\right) \\ &= h + \frac{4M(M-m)u^2}{(M+m)^2 g} \end{aligned}$$

as required.

Marks

B1: Maximum height where velocity is zero

M1: Calculating KE of B before and after collision

A1 ft: Calculating gain in KE correctly (need not be simplified)

M1: Height reached above h found by using: gain in GPE at max height = extra KE gained from collision

A1 ft: $mgH = \frac{1}{2}mv_B^2 - \frac{1}{2}mu^2$

M1: Expanding fractions and simplifying

A1 cso (AG): Calculating max height in form given in question

Alternative: "suvat" approach

B1: Speed is zero at maximum height

M1 A1: Calculating displacement above point of collision

M1 A1: Calculating drop from initial height to point of collision (NB: The candidate must be clear that this is what they are attempting to do to be awarded these marks)

M1 A1 cso: Correctly (and clearly) combining the height differences to find the required answer

[Total for finding max height of B: 7 marks]

Question 11

A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A. Show that

$$3\sin\alpha = 4\sin\beta\,,$$

where α and β are the angles PAB and PBA, respectively.

We begin by drawing a diagram of the situation, showing the forces involved (the tension in the string, which is the same at A and B since the peg is smooth, and the weight of the bar acting through G). Clearly $BG = 4d$, which we have shown as well.

We have indicated the angles α, β and ϕ as defined in the question, and have also introduced the angle θ as angle AGP. The point M is the foot of the perpendicular from P to AB, which is used in some of the methods of solution.

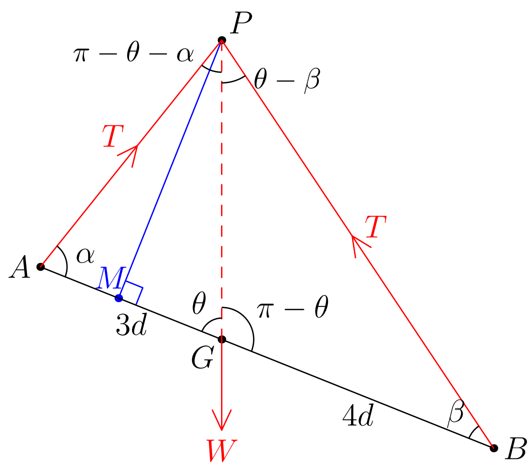


Diagram 1: Using θ for angle between rod and vertical

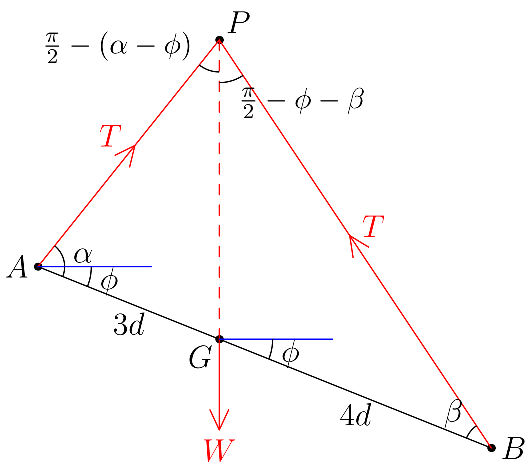


Diagram 2: Using ϕ for angle between rod and horizontal

Note that we have drawn the sketch with the weight passing through P. This must be the case: both tensions pass through P and the system is in equilibrium. So taking moments around P shows that W times the distance of the line of force of W from P must be zero, so that W acts through P.

The simplest way of showing that $3\sin\alpha = 4\sin\beta$ is to take moments about G:

$$\mathscr{M}(\widehat{G}) \qquad T.3d\sin\alpha - T.4d\sin\beta = 0$$

so that $3\sin\alpha = 4\sin\beta$.

An alternative approach is to apply the sine rule to the triangles PAG and PBG and resolve horizontally; this, though, is a somewhat longer-winded method.

Marks

[Mark for weight through P is awarded later if relevant]

M1: Clear diagram

M1: Taking moments about A, B or G; forces must appear (correctly) in the resulting equation to be awarded this mark

A1: Correct moment for at least one force

A1 cso (AG): Deducing stated equation

[Total for this part: 4 marks]

> *Given that $\cos\beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan\frac{1}{7}$.*

From $\cos\beta = \frac{4}{5}$, we deduce $\sin\beta = \frac{3}{5}$, and hence $\sin\alpha = \frac{4}{3}\sin\beta = \frac{4}{5}$. Thus $\cos\alpha = \pm\frac{3}{5}$, and since α is acute, $\cos\alpha = \frac{3}{5}$.

There are numerous ways of finding the length of the string, l, in terms of d. We present a few approaches here.

Approach 1: Show that $\angle APB = \frac{\pi}{2}$

To find the length of the string, we first find the angle APB.

One method is to note that $\angle APB = \pi - \alpha - \beta$, so

$$\begin{aligned}\sin(\pi-\alpha-\beta) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \tfrac{4}{5}\times\tfrac{4}{5} + \tfrac{3}{5}\times\tfrac{3}{5}\\ &= 1,\end{aligned}$$

so $\angle APB = \frac{\pi}{2}$ and the triangle APB is right-angled at P.

Alternatively, as $\sin\alpha = \cos\beta$, we must have $\alpha + \beta = \frac{\pi}{2}$, so $\angle APB = \frac{\pi}{2}$.

Thus $AP = AB\cos\alpha = 7d.\frac{3}{5} = \frac{21}{5}d$ and $BP = AB\sin\alpha = 7d.\frac{4}{5} = \frac{28}{5}d$, so the string has length $(\frac{21}{5} + \frac{28}{5})d = \frac{49}{5}d$.

Approach 2: Trigonometry with the perpendicular from P

In the triangle APB, we draw a perpendicular from P to AB, meeting AB at M. Then $PM = AP\sin\alpha = BP\sin\beta$. (This can also be shown directly by applying the sine rule: $AP/\sin\beta = BP/\sin\alpha$.)

We also have $AB = AM + BM = AP\cos\alpha + BP\cos\beta = 7d$.

Now using our known values of $\sin\alpha$, etc., these equations become $\frac{4}{5}AP = \frac{3}{5}BP$ so that $BP = \frac{4}{3}AP$, and $\frac{3}{5}AP + \frac{4}{5}BP = 7d$.

Combining these gives $\frac{3}{5}AP + \frac{16}{15}AP = 7d$, so $AP = \frac{21}{5}d$ and hence $BP = \frac{28}{5}d$ and $l = AP + BP = \frac{49}{5}d$.

Approach 3: Cosine rule

We apply the cosine rule to the triangle APB, and we write $x = AP$ and $y = BP$ for simplicity. This gives us

$$\begin{aligned}x^2 &= AB^2 + y^2 - 2AB.y\cos\beta \qquad \text{or}\\ y^2 &= AB^2 + x^2 - 2AB.x\cos\alpha.\end{aligned}$$

There are different ways of continuing from here. The most straightforward is probably to begin by showing that $BP = \frac{4}{3}AP$ or $y = \frac{4}{3}x$ as in Approach 2. This then simplifies the two equations to give

$$\begin{aligned}(\tfrac{3}{4}y)^2 &= (7d)^2 + y^2 - 2(7d).y.\tfrac{4}{5} \qquad \text{or}\\ (\tfrac{4}{3}x)^2 &= (7d)^2 + x^2 - 2(7d).x.\tfrac{3}{5}.\end{aligned}$$

We can then expand and rearrange these quadratics to get

$$\begin{aligned}\tfrac{7}{16}y^2 - \tfrac{56}{5}dy + 49d^2 &= 0 \qquad \text{or}\\ \tfrac{7}{9}x^2 + \tfrac{42}{5}dx - 49d^2 &= 0.\end{aligned}$$

Dividing by 7 and clearing fractions (multiplying by 80 and 45 respectively) gives

$$\begin{aligned}5y^2 - 128dy + 560d^2 &= 0 \qquad \text{or}\\ 5x^2 + 54dx - 315d^2 &= 0.\end{aligned}$$

These quadratics turn out to factorise as

$$\begin{aligned}(y - 20d)(5y - 28d) &= 0 \qquad \text{or}\\ (x + 15d)(5x - 21d) &= 0.\end{aligned}$$

The first equation gives two possibilities: $y = 20d$ or $y = \frac{28}{5}d$, whereas the second only gives one: $x = \frac{21}{5}d$.

For the first equation, $y = 20d$ would imply $x = \frac{3}{4}y = 15d$, but then we would have

$$\cos\alpha = \frac{AB^2 + x^2 - y^2}{2AB.x} = \frac{49d^2 + 225d^2 - 400d^2}{14d.15d} < 0,$$

which is not possible as α is acute.

So we must have $x = \frac{21}{5}d$ and $y = \frac{28}{5}d$, and hence $l = x + y = \frac{49}{5}d$.

Approach 4: Sine rule

Using the sine rule on the triangle APB, we have

$$\frac{AB}{\sin(\pi - \alpha - \beta)} = \frac{AP}{\sin\beta} = \frac{BP}{\sin\alpha}.$$

We use $\sin(\pi - \phi) = \sin\phi$ and the addition (compound angle) formula to write

$$\begin{aligned}\sin(\pi - \alpha - \beta) &= \sin(\alpha + \beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \tfrac{4}{5} \times \tfrac{4}{5} + \tfrac{3}{5} \times \tfrac{3}{5}\\ &= 1,\end{aligned}$$

so that the sine rule becomes

$$\frac{7d}{1} = \frac{AP}{3/5} = \frac{BP}{4/5},$$

giving $AP = \frac{21}{5}d$, $BP = \frac{28}{5}d$ and hence $l = AP + BP = \frac{49}{5}d$.

Marks

B1: Determining $\sin\beta$
M1: Using first part to determine $\sin\alpha$
A1 cao: Correct $\sin\alpha$ *and* $\cos\alpha$

Approach 1 (Show that $\angle APB = \frac{\pi}{2}$*):*
M1 M1 A1: Showing $\angle APB = \frac{\pi}{2}$
M1 A1: Finding one of AP *or* BP
A1 cao: Determining the length of the string

Approach 2 (Trigonometry with the perpendicular from P*):*
M1 A1 A1: Finding the equations
M1 A1: Finding one of AP *or* BP
A1 cao: Determining the length of the string

Approach 3 (cosine rule):
[If using this method, only award maximum 2 marks for finding sines and cosines above]
M1 A1: Showing that $PB = \frac{4}{3}AP$
M1 (dep) A1 A1: Applying the cosine rule to get a quadratic equation
M1 A1 cao: Deducing the correct root to use (this is not obvious in one of the two cases) and determining length of string

Approach 4 (sine rule):
M1 A1: Showing that $PB = \frac{4}{3}AP$ *or equivalent*
M1: Using sine rule
M1: Expanding $\sin(\pi - \alpha - \beta)$ *and substituting angles to simplify*
M1 A1 cao: Determining correct length of string
[Total for finding length of string: 9 marks]

We now find ϕ, the angle of inclination of the bar to the horizontal. Referring to the above diagrams, we have $\phi = \frac{\pi}{2} - \theta$, so $\tan\phi = \cot\theta$.

Here again are several approaches to this problem.

Approach 1: Resolving forces horizontally and vertically

We resolve horizontally to get

$$\mathscr{R}(\rightarrow) \qquad T\sin(\pi - \theta - \alpha) - T\sin(\theta - \beta) = 0.$$

Therefore we get

$$\sin(\theta + \alpha) = \sin(\theta - \beta).$$

We now use the addition formula for sine to expand these, and then substitute in our values for $\sin\alpha$, etc., giving:

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = \sin\theta\cos\beta - \cos\theta\sin\beta$$

so

$$\tfrac{3}{5}\sin\theta + \tfrac{4}{5}\cos\theta = \tfrac{4}{5}\sin\theta - \tfrac{3}{5}\cos\theta$$

giving

$$\tfrac{7}{5}\cos\theta = \tfrac{1}{5}\sin\theta.$$

Dividing by $\cos\theta$ now gives $\cot\theta = \frac{1}{7}$, hence the angle made with the horizontal is given by $\tan\phi = \frac{1}{7}$, yielding $\phi = \arctan\frac{1}{7}$ as required.

Alternatively, using ϕ instead of θ in the original equations, we get

$$\mathscr{R}(\rightarrow) \qquad T\sin(\tfrac{\pi}{2}+\phi-\alpha) - T\sin(\tfrac{\pi}{2}-\phi-\beta) = 0,$$

which simplifies (on dividing by $T \neq 0$ and using $\sin(\frac{\pi}{2}-x)=\cos x$) to

$$\cos(\alpha-\phi) - \cos(\phi+\beta) = 0.$$

The rest of the argument follows as before.

Approach 2: Resolving forces parallel and perpendicular to the rod

We resolve parallel to the rod to get

$$\mathscr{R}(\searrow) \qquad T\cos\alpha + W\cos\theta - T\cos\beta = 0$$

and perpendicular to the rod to get

$$\mathscr{R}(\nearrow) \qquad T\sin\alpha - W\sin\theta + T\sin\beta = 0.$$

Rearranging these gives:

$$\begin{aligned} W\cos\theta &= -T\cos\alpha + T\cos\beta \\ W\sin\theta &= T\sin\alpha + T\sin\beta. \end{aligned}$$

Dividing these equations gives

$$\begin{aligned} \cot\theta &= \frac{-\cos\alpha + \cos\beta}{\sin\alpha + \sin\beta} \\ &= \frac{-\frac{3}{5}+\frac{4}{5}}{\frac{4}{5}+\frac{3}{5}} \\ &= \tfrac{1}{7}. \end{aligned}$$

Since $\tan\phi = \cot\theta$, as we noted above, we have $\tan\phi = \frac{1}{7}$, so $\phi = \arctan\frac{1}{7}$ as required.

Approach 3: Dropping a perpendicular

In the triangle APB, we draw a perpendicular from P to AB, meeting AB at M. Then $PM = AP\sin\alpha = \frac{21}{5}d.\frac{4}{5} = \frac{84}{25}d$ and $AM = AP\cos\alpha = \frac{21}{5}d.\frac{3}{5} = \frac{63}{25}d$. As $AG = 3d$, it follows that $MG = AG - AM = \frac{12}{25}d$.

Then (see the diagram above) we have $\tan\theta = PM/MG = \frac{84}{25}d/\frac{12}{25}d = 7$ so that $\cot\theta = \tan\phi = \frac{1}{7}$, giving $\phi = \arctan\frac{1}{7}$ as required.

Approach 4: PG bisects $\angle APB$

As in approach 1 above, we resolve horizontally (using diagram 2) to get

$$T\sin(\tfrac{\pi}{2} - (\alpha-\phi) - T\sin(\tfrac{\pi}{2}-\phi-\beta) = 0.$$

Since both of the angles involved here are acute (as the triangle BGP has an obtuse angle at G), they must be equal, giving $\alpha - \phi = \phi + \beta$, so that $2\phi = \alpha - \beta$.

Hence we have

$$\begin{aligned}\cos 2\phi &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \tfrac{3}{5}\cdot\tfrac{4}{5} + \tfrac{4}{5}\cdot\tfrac{3}{5} \\ &= \tfrac{24}{25}.\end{aligned}$$

We therefore deduce using $\cos 2\phi = 2\cos^2\phi - 1$ that $\cos^2\phi = \frac{49}{50}$ and $\sin^2\phi = \frac{1}{50}$. It follows that $\tan^2\phi = \sin^2\phi/\cos^2\phi = \frac{1}{49}$, giving $\tan\phi = \frac{1}{7}$ (the positive root as ϕ is acute) or $\phi = \arctan\frac{1}{7}$ as required.

Marks

Approach 1 (resolving forces horizontally and vertically):
B1: Determining angles of forces to horizontal or vertical
M1 A1: Resolving forces horizontally
M1 A1: Expanding using the compound angle formulæ
M1 A1 (AG): Rearranging to find $\tan\theta$ or equivalent and the required answer

Approach 2 (resolving parallel and perpendicular):
M1 A1: Resolving forces parallel to rod correctly
M1 A1: Resolving forces perpendicular to rod correctly
M1 A1: Dividing equations to find $\cot\theta$ or equivalent
A1 cao (AG): Reaching given answer $\arctan\frac{1}{7}$

Approach 3 (dropping a perpendicular):
M1 A1: Determining AM
B1: Stating $\tan\theta = GM/PM$
M1 A1 A1: Calculating GM and PM
A1 cao (AG): Reaching the given answer

Approach 4 (dropping a perpendicular):
M1: Deducing angles in triangle in terms of α, β and ϕ or θ
M1 A1: Showing vertical bisects angle at P
M1 A1: Using compound angle formula to deduce $\cos 2\phi$
M1 A1 (AG): Using the double angle formulæ to deduce the given answer
[Total for finding angle with horizontal: 7 marks]

Question 12

I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

I can sell one ticket to each person as long as I have a £1 coin when the single person with a £2 coin arrives, which will be the case as long as they are not the first person in the queue. Thus the probability is

$$1 - \frac{1}{m+1} = \frac{m}{m+1}.$$

Marks
M1: Argument
A1 cao: Answer; award both marks even if justification is very weak in this part.
[Total for part (i): 2 marks]

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

This time, I can sell to all the people as long as I have one £1 coin when the first £2 coin is given to me and I have received at least two £1 coins (in total) by the time the second £2 coin is offered.

So we consider the first three people in the queue and the coin they bring; in the table below, "any" means that either coin could be offered at this point. (This called also be represented as a tree diagram, of course.) The probabilities in black are those of success, the ones in red are for the cases of failure. Only one or the other of these needs to be calculated.

1st	2nd	3rd	Success?	Probability
£1	£1	any	yes	$\dfrac{m}{m+2} \times \dfrac{m-1}{m+1} = \dbinom{m}{2} \Big/ \dbinom{m+2}{2}$
£1	£2	£1	yes	$\dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{m-1}{m} = \dbinom{m-1}{1} \Big/ \dbinom{m+2}{2}$
£1	£2	£2	no	$\dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{1}{m} = 1 \Big/ \dbinom{m+2}{2}$
£2	any	any	no	$\dfrac{2}{m+2} = \dbinom{m+1}{1} \Big/ \dbinom{m+2}{2}$

To determine the probabilities in the table, there are two approaches. The first is to find the probability that the kth person brings the specified coin given the previous coins which have been brought; this is the most obvious method when this is drawn as a tree diagram. The second approach is to count the number of possible ways of arranging the remaining coins and to divide it by the total number of possible arrangements of the $m+2$ coins, which is $\binom{m+2}{2} = \frac{1}{2}(m+2)(m+1)$.

Therefore the probability of success is

$$\frac{m(m-1)}{(m+2)(m+1)} + \frac{2(m-1)}{(m+2)(m+1)} = \frac{(m+2)(m-1)}{(m+2)(m+1)} = \frac{m-1}{m+1}.$$

Alternatively, we could calculate the probability of failure (adding up the probabilities in red) and subtract from 1 to get

$$1 - \frac{2}{(m+2)(m+1)} - \frac{2}{m+2} = 1 - \frac{2}{m+2}.\frac{1+(m+1)}{m+2} = 1 - \frac{2}{m+2} = \frac{m-1}{m+2}.$$

Marks

M1: Explaining the situations under which we will have success or those under which we will have failure (can be implied by table or tree diagram or similar)

M1: Determining the probability for the case £1, £1 ... (or £1, £2, £2, ... if considering failure)

A1 cao: ... correctly

M1: Determining the probability in the case £1, £2, £1 ... (or £2, ... if considering failure)

A1: ... correctly

M1: Rejecting remaining cases

M1: Adding probabilities and simplifying (and subtracting from 1 if considering failure)

A1 cso (AG): Reaching given answer

[Total for part (ii): 8 marks]

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

This time, it turns out that we need to consider the first five people in the queue to distinguish the two cases which begin with £1, £1, £2, £2; the rest of the method is essentially the same as in part (ii).

1st	2nd	3rd	4th	5th	Success?	Probability
£1	£1	£1	any	any	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{m-2}{m+1} = \binom{m}{3} \Big/ \binom{m+3}{3}$
£1	£1	£2	£1	any	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{m-2}{m} = \binom{m-1}{2} \Big/ \binom{m+3}{3}$
£1	£1	£2	£2	£1	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1}$

						$= \binom{m-2}{1} \Big/ \binom{m+3}{3}$
£1	£1	£2	£2	£2	no	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{2}{m} \times \frac{1}{m-1}$
						$= 1 \Big/ \binom{m+3}{3}$
£1	£2	£1	£1	any	yes	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{m-2}{m}$
						$= \binom{m-1}{2} \Big/ \binom{m+3}{3}$
£1	£2	£1	£2	£1	yes	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1}$
						$= \binom{m-2}{1} \Big/ \binom{m+3}{3}$
£1	£2	£1	£2	£2	no	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{2}{m} \times \frac{1}{m-1}$
						$= 1 \Big/ \binom{m+3}{3}$
£1	£2	£2	any	any	no	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{2}{m+1} = \binom{m}{1} \Big/ \binom{m+3}{3}$
£2	any	any	any	any	no	$\frac{3}{m+3} = \binom{m+2}{2} \Big/ \binom{m+3}{3}$

(Alternatively, the four cases beginning £1, £2 can be regarded as £1, £2 followed by 2 people with £2 coins and $m-1$ people with £1 coins, bringing us back into the case of part (ii). So the probability of success in these cases is $\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-2}{m}$, where the final fraction comes from the result of part (ii).)

Therefore the probability of success is

$$\frac{1}{(m+3)(m+2)(m+1)}\big(m(m-1)(m-2) + 3(m-1)(m-2) + 6(m-2) + 3(m-1)(m-2) + 6(m-2)\big)$$

$$= \frac{m-2}{(m+3)(m+2)(m+1)}\big(m(m-1) + 6(m-1) + 12\big)$$

$$= \frac{m-2}{(m+3)(m+2)(m+1)}(m^2 + 5m + 6)$$

$$= \frac{m-2}{m+1}.$$

Similarly, the probability of success can be calculated by considering the probability of failure:

the probability of success is therefore

$$\begin{aligned}
1 - \frac{6}{(m+3)(m+2)(m+1)} - \frac{6}{(m+3)(m+2)(m+1)} - \frac{6m}{(m+3)(m+2)(m+1)} - \frac{3}{m+3}& \\
= 1 - \frac{12 + 6m + 3(m+1)(m+2)}{(m+3)(m+2)(m+1)}& \\
= 1 - \frac{3m^2 + 15m + 18}{(m+3)(m+2)(m+1)}& \\
= 1 - \frac{3(m+2)(m+3)}{(m+3)(m+2)(m+1)}& \\
= 1 - \frac{3}{m+1}& \\
= \frac{m-2}{m+1}.&
\end{aligned}$$

There seems to be a pattern in these results, and one might conjecture that the probability of being able to sell one ticket to each person in the general case $m \geqslant n$ is $\frac{m+1-n}{m+1}$. This turns out to be correct, though the proof uses significantly different ideas from those used above.

Marks

M1: Setting up an effective method, either a table considering first five people in queue or a correctly structured tree diagram covering all necessary cases

M1 indep: Correct method for any case (can award for any correct case even if fewer than five people considered)

A1: Correct answer for £1, £1, £1 case

A1: Correct answer for £1, £1, £2, £1 case

A1: Correct answer for £1, £1, £2, £2, £1 case

M1: Either referring back to part (ii) or considering remaining two good cases

A1: Correct answer for this case / these cases

M1: Rejecting remaining cases

A1 ft: Correct probability of success (unsimplified sufficient for this mark)

A1 cso (AG): Reaching given answer

(A similar distribution of marks is awarded in the case that the candidate adds the probabilities of failure and calculates $1 - \mathrm{P}(\text{failure})$.)
[Total for part (iii): 10 marks]

Question 13

In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f *given by*

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \textit{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

We know that $\int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = 1$, so we begin by performing this integration to determine k.

We have

$$\begin{aligned}\int_{-a}^{0} 2k\,\mathrm{d}x + \int_0^2 k\sqrt{4-x^2}\,\mathrm{d}x &= \big[2kx\big]_{-a}^{0} + k\big[2\arcsin\tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4-x^2}\,\big]_0^2 \\ &= 2ak + k\big((2\arcsin 1 + 0) - (2\arcsin 0 + 0)\big) \\ &= 2ak + k\pi \\ &= k(2a+\pi) \\ &= 1,\end{aligned}$$

so $k = 1/(2a+\pi)$.

We can now work out the mean of X; we work in terms of k until the very end to avoid ugly calculations. We can integrate the expression $x\sqrt{4-x^2} = kx(4-x^2)^{\frac{1}{2}}$ either using inspection (as we do in the following) or the substitution $u = 4-x^2$, giving $\mathrm{d}u/\mathrm{d}x = -2x$, so that the integral becomes $k\int_4^0 -\frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u = k\big[-\frac{1}{3}u^{\frac{3}{2}}\big]_4^0 = \frac{8}{3}k$.

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\mathrm{f}(x)\,\mathrm{d}x \\ &= \int_{-a}^{0} 2kx\,\mathrm{d}x + \int_0^2 kx\sqrt{4-x^2}\,\mathrm{d}x \\ &= \big[kx^2\big]_{-a}^{0} + \big[-\tfrac{k}{3}(4-x^2)^{\frac{3}{2}}\big]_0^2 \\ &= (0-ka^2) + (0-(-\tfrac{k}{3}\times 4^{\frac{3}{2}})) \\ &= -ka^2 + \tfrac{8}{3}k \\ &= \frac{\frac{8}{3}-a^2}{2a+\pi} \\ &= \frac{8-3a^2}{3(2a+\pi)}.\end{aligned}$$

Marks

M1: Using $\int \mathrm{f}(x)\,\mathrm{d}x = 1$ to find k
M1: Splitting integral and integrating each part
A1: Correct evaluation of definite integral
A1 cao: Correct k

For the rest of the question, follow through an incorrect k except in "cao" or "cso" marks

M1: Splitting correct $\mathrm{E}(X)$ integral into two parts correctly
M1: Integrating $x\sqrt{4-x^2}$ to get something of the form $A(4-x^2)^{\frac{3}{2}}$
A1: Correctly integrating both terms (can still be in terms of k)
A1: Correctly evaluating integral (can still be in terms of k)
A1 cao: Correct integral in terms of a alone
[Total for part (i): 9 marks]

(ii)	*Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.*

We have $d < 0$ if and only if $\mathrm{P}(X > 0) < \mathrm{P}(X > d) = \frac{1}{10}$, so we consider $\mathrm{P}(X > 0)$. Using the above integration (or noting that X is uniform for $x < 0$), we have

$$\begin{aligned}
\mathrm{P}(X > 0) &= 1 - \mathrm{P}(X < 0) \\
&= 1 - 2ak \\
&= 1 - \frac{2a}{2a+\pi} \\
&= \frac{\pi}{2a+\pi}.
\end{aligned}$$

Therefore $\mathrm{P}(X > 0) < \frac{1}{10}$ if and only if

$$\frac{\pi}{2a+\pi} < \frac{1}{10};$$

that is $10\pi < 2a + \pi$, or $2a > 9\pi$. Putting these together gives $d < 0$ if and only if $2a > 9\pi$.

In this case, as $d < 0$, we have

$$\mathrm{P}(X > d) = 1 - \mathrm{P}(X < d) = 1 - 2k(d - (-a)),$$

so $1 - 2k(d + a) = \frac{1}{10}$, so $d + a = \frac{9}{10}/2k$, giving

$$\begin{aligned}
d &= \frac{9}{20k} - a \\
&= \frac{9(2a+\pi)}{20} - a \\
&= \frac{9\pi - 2a}{20}.
\end{aligned}$$

Note that, since $2a > 9\pi$, this gives us $d < 0$ as we expect.

An alternative approach is to calculate the cumulative distribution function first. We have

$$\mathrm{F}(x) = \begin{cases} 0 & x < -a \\ 2k(x+a) & -a \leqslant x < 0 \\ k\big(2a + 2\arcsin\frac{1}{2}x + \frac{1}{2}x\sqrt{4-x^2}\big) & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

(though only the part with $-a \leqslant x \leqslant 0$ is actually needed).

Then we solve $\mathrm{F}(d) = \frac{9}{10}$. If it turns out that $d < 0$, then we have $\mathrm{F}(d) = 2k(d + a) = \frac{9}{10}$, which rearranges to give $d = (9\pi - 2a)/20$ as above. Now if $2a > 9\pi$, then $(9\pi - 2a)/20 < 0$ so that $F((9\pi - 2a)/20) = \frac{9}{10}$ and $d < 0$, as required.

Marks

M1: Converting $d < 0$ into the condition $\mathrm{P}(X > 0) < \frac{1}{10}$

M1: Using uniformity or other approach to find $\mathrm{P}(X > 0)$

A1: Finding $\mathrm{P}(X > 0)$ in terms of a alone

M1: Using this in the inequality $\mathrm{P}(X > 0) < \frac{1}{10}$ to find a condition on a

A1 cso: Deducing that $d < 0$ if given inequality $2a > 9\pi$ holds; award also if the reverse implication is proven instead

M1: Finding an expression for $\mathrm{P}(X > d)$ or $\mathrm{P}(X < d)$ in this case

A1 cso: Find d in terms of a correctly

Alternative approach via cdf

M1 A1: Find cdf for $-a < x < 0$

M1: Rearranging $F(d) = \frac{9}{10}$ to get d in terms of a, assuming $d < 0$ explicitly

A1: Reaching $d = (9\pi - 2a)/20$

M1 A1: This works if or only if $2a > 9\pi$ (at least one direction of argument)

A1 cso: Correct direction of argument: if $2a > 9\pi$ then $d < 0$

[Total for part (ii): 7 marks]

(iii) *Given that $d = \sqrt{2}$, find a.*

We note that now $d > 0$, so we have to integrate to find a explicitly. We get

$$
\begin{aligned}
\mathrm{P}(X > \sqrt{2}) = \int_{\sqrt{2}}^{2} k\sqrt{4 - x^2}\,\mathrm{d}x &= \left[2\arcsin\tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4 - x^2}\,\right]_{\sqrt{2}}^{2} \\
&= k\big((2\arcsin 1 + 0) - (2\arcsin\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\sqrt{4 - 2})\big) \\
&= k(\pi - \tfrac{\pi}{2} - 1) \\
&= k(\tfrac{\pi}{2} - 1) \\
&= \frac{\frac{\pi}{2} - 1}{2a + \pi} \\
&= \tfrac{1}{10}.
\end{aligned}
$$

Thus $10(\frac{\pi}{2} - 1) = 2a + \pi$, so that $2a = 4\pi - 10$, giving our desired result: $a = 2\pi - 5$.

Alternatively, one could calculate $\mathrm{P}(X < \sqrt{2})$ in the same manner and find a such that this equals $\frac{9}{10}$.

As a check, it is clear that $2a = 4\pi - 10 < 9\pi$, so $d \geqslant 0$ from part (ii).

Marks

M1: Integrating to find $\mathrm{P}(X > \sqrt{2})$ or $\mathrm{P}(X < \sqrt{2})$ explicitly

A1: Reaching $k(\frac{\pi}{2} - 1)$ (possibly with k substituted), or $1 - k(\frac{\pi}{2} - 1)$ if $\mathrm{P}(X > \sqrt{2})$ is calculated

M1: Substituting k and solving to find a

A1 cso: Correct a

[Total for part (iii): 4 marks]