GCE

Mathematics

Unit **4726**: Further Pure Mathematics 2

Advanced GCE

Mark Scheme for June 2018

OCR (Oxford Cambridge and RSA) is a leading UK awarding body, providing a wide range of qualifications to meet the needs of candidates of all ages and abilities. OCR qualifications include AS/A Levels, Diplomas, GCSEs, Cambridge Nationals, Cambridge Technicals, Functional Skills, Key Skills, Entry Level qualifications, NVQs and vocational qualifications in areas such as IT, business, languages, teaching/training, administration and secretarial skills.

It is also responsible for developing new specifications to meet national requirements and the needs of students and teachers. OCR is a not-for-profit organisation; any surplus made is invested back into the establishment to help towards the development of qualifications and support, which keep pace with the changing needs of today's society.

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

Annotations and abbreviations

Annotation available	**Meaning**
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
soi	Seen or implied
www	Without wrong working

Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be ‘follow through’. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g **Rules for replaced work**

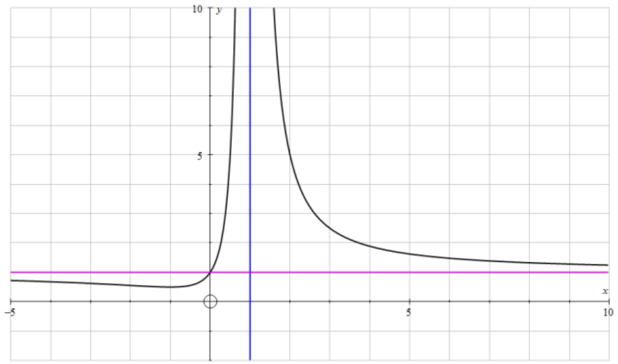
If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate’s data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

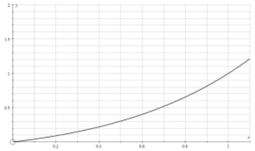
Note that a miscopy of the candidate’s own working is not a misread but an accuracy error.

Question			Answer	Marks	Guidance
1	(i)		$x = 1$ $y = 1$	B1 B1	-1 for any extra if 2 scored
				[2]	
	(ii)		$y = \frac{x^2+1}{(x-1)^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1)^2.2x - 2(x^2+1)(x-1)}{(x-1)^4}$ $= 0$ when $(x-1)^2.2x = 2(x^2+1)(x-1)$ $\Rightarrow x(x-1) = x^2+1 \Rightarrow x = -1$ or $x = 1$, reject $\Rightarrow y = \frac{1+1}{(-2)^2} = \frac{1}{2}$ i.e. $\left(-1, \frac{1}{2}\right)$ Justification that this is a minimum $\Rightarrow y \geq \frac{1}{2}$ Alternatively: $y = k \Rightarrow$ quad in $x \Rightarrow x^2(k-1) - 2kx + (k-1) = 0$ real roots $\Rightarrow (2k)^2 \geqslant 4(k-1)^2$ $\Rightarrow k = y \geqslant \frac{1}{2}$ $\Rightarrow$ Minimum is at $y = \frac{1}{2}$ $x = -1$	M1 A1 A1 A1 M1 A1	and set = 0 Correct numerator Correct x Correct y By any means SC B1 $y \geqslant 0.5$ M1 A1 correct quadratic equation M1 considering discriminant of their three term quadratic - allow > A1 A1 for y A1 for x
				[6]	

Question			Answer	Marks	Guidance
	(iii)			B1 B1	For rhs, approaching $y = 1$ from above and asymptotes labelled or indicated by scales on axes. For lhs including (0, 1) and approaching $y = 1$ from below
				[2]	

Question			Answer	Marks	Guidance
2	(i)		$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = \left(\frac{e^{2x} + 2 + e^{-2x}}{2}\right) - 1$ $= \left(\frac{e^{2x} + e^{-2x}}{2}\right) = \cosh 2x$	M1 A1	Using exponential form of cosh, squaring and expanding. **AG**
				[2]	
	(ii)		$\int_0^1 \cosh^2 3x \, dx = \int_0^1 \frac{1 + \cosh 6x}{2} \, dx = \left[\frac{x}{2} + \frac{\sinh 6x}{12}\right]_0^1$ $= \frac{1}{2} + \frac{1}{12}\sinh 6$	M1 A1 A1	Use of double angle and integrate
				[3]	

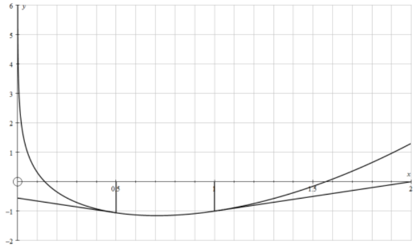
Question			Answer	Marks	Guidance
3			$y = \cosh x - 2\sinh 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x - 4\cosh 2x$	M1	Diffn and set = 0
			$= 0$ when $\sinh x = 4\cosh 2x$	A1	
			$\cosh 2x = 1 + 2\sinh^2 x$	M1	Use of double angle formula to obtain a quadratic in sinh$x$
			$\Rightarrow \sinh x = 4 + 8\sinh^2 x \Rightarrow 8\sinh^2 x - \sinh x + 4 = 0$	A1	Correct quadratic
			This equation has no roots as $"b^2 - 4ac" < 0 \quad (= -127)$		
			$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ for all x.	A1	**ft** Correct and complete conclusion
			or $\Rightarrow$ no turning points Alternatively: Use of exponentials leads to $4\mathrm{e}^{4x} - \mathrm{e}^{3x} + \mathrm{e}^{x} + 4 = 0$ can only earn 3 marks		M1 change to exponentials M1 differentiate A1 correct and simplified quartic
				[5]	

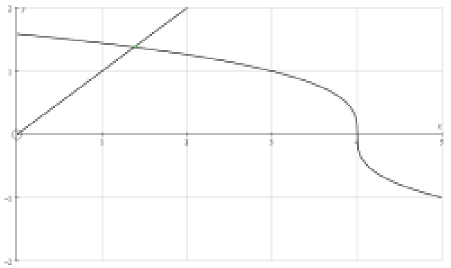
Question			Answer	Marks	Guidance
4	(i)		$I_n = \int_0^1 x^n e^{x-1} dx \quad u = x^n \Rightarrow du = nx^{n-1} dx$ $dv = e^{x-1} dx \Rightarrow v = e^{x-1}$ $= \left[x^n e^{x-1} \right]_0^1 - n\int_0^1 x^{n-1} e^{x-1} dx$ $= 1 - nI_{n-1}$	**M1** **A1** **A1**	Correct *u* and *v* Term at beginning - ignore limits All correct
				[3]	
	(ii)		$I_4 = 1 - 4I_3 = 1 - 4(1 - 3I_2) = -3 + 12I_2$ $= -3 + 12(1 - 2I_1) = 9 - 24I_1$ $I_1 = \int_0^1 xe^{x-1} dx = \left[xe^{x-1} \right]_0^1 - \int_0^1 e^{x-1} dx = 1 - \left(1 - \frac{1}{e}\right)$ $= \frac{1}{e} \Rightarrow I_4 = 9 - \frac{24}{e}$	**M1** **A1** **B1** **A1**	Use of formula - achieving I_2 from either end will earn this mark Do not accept I_0
				[4]	
	(iii)		$y = x^n e^{x-1} \Rightarrow \frac{dy}{dx} = x^n e^{x-1} + nx^{n-1} e^{x-1} = x^{n-1} e^{x-1} (n + x)$  (For all n) the curve passes through (0, 0) and (1,1). Justifying $I_n < 1$ Justifying $I_n > 0$	**B1** **B1** **B1** **B1** **B1**	Finding gradient Sketch - smooth curve monotonically increasing above *x* axis. Can be seen on graph. Implication that I_n is the area under the graph or within square Positive *y* or positive gradient throughout so > 0. Or a clesrly shaded positive region on graph
				[5]	

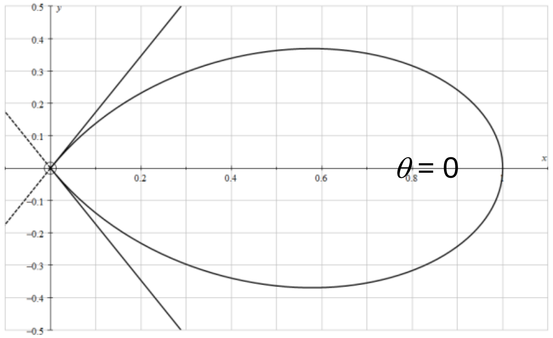
Question			Answer	Marks	Guidance
5			$I=\int_0^{\pi/2}\frac{1}{\sin x+1}\,dx$		
			Set $t=\tan\frac{1}{2}x$; $\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{1}{2}x\Rightarrow dx=\frac{2}{\left(1+t^2\right)}dt$	**B1**	dx
			$\tan x=\frac{2t}{1-t^2}\Rightarrow\sin x=\frac{2t}{1+t^2}\Rightarrow 1+\sin x=\frac{\left(1+t\right)^2}{1+t^2}$	**B1**	1+sinx
			$\Rightarrow I=\int\frac{1+t^2}{\left(1+t\right)^2}\frac{2}{\left(1+t^2\right)}dt=\int\frac{2}{\left(1+t\right)^2}dt$	**B1**	Dep on both previous B. correct integrand
			$x=0\Rightarrow t=0.\quad x=\frac{\pi}{2}\Rightarrow t=1$	**M1**	Integrate using correct limits
			$\Rightarrow I=\left[\frac{-2}{\left(1+t\right)}\right]_0^1=\frac{-2}{2}-\frac{-2}{1}=1$	**A1**	Ans
				[5]	

Question			Answer	Marks	Guidance
6	(i)		$y=\tan^{-1}(3x)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{1+9x^2}$	**B1** **B1**	numerator denominator
				[2]	
	(ii)		$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{1+9x^2}\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-3(18x)\frac{1}{\left(1+9x^2\right)^2}$ $=\frac{-54x}{\left(1+9x^2\right)^2}=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{-18x}{\left(1+9x^2\right)}$ $\Rightarrow\left(1+9x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18x\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\left(1+9x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+18x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18\frac{\mathrm{d}y}{\mathrm{d}x}+18x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\Rightarrow\left(1+9x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+36x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18\frac{\mathrm{d}y}{\mathrm{d}x}=0$	**M1** **M1** **M1** **A1**	Correct diffn *their* fn of the form $\frac{a}{\left(1+bx^2\right)}$ from (i) Multiply to get eqn Diffn again www **AG**
			Alternative: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{1+9x^2}\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-54x}{\left(1+9x^2\right)^2}$ $\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{54}{\left(1+9x^2\right)^3}\left(27x^2-1\right)$ $\Rightarrow\left(1+9x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+36x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\frac{54}{\left(1+9x^2\right)^2}\left(27x^2-1\right)-36x\frac{54x}{\left(1+9x^2\right)^2}+\frac{54}{1+9x^2}$ $=\frac{54}{\left(1+9x^2\right)^2}\left(27x^2-1-36x^2+1+9x^2\right)=0$		M1 Diffn *their* fn of the form $\frac{a}{\left(1+bx^2\right)}$ from (i) M1 Diffn again using product or quotient rule M1 using their results in given expression A1 www
				[4]	

Question			Answer	Marks	Guidance
	(iii)		$y_0 = 0, \quad y'_0 = 3, \quad y''_0 = 0, \quad y'''_0 = -54$ $\Rightarrow y = 3x - 9x^3$	**B1** **M1** **A1**	All four soi Correct formula SC3: Correct answer, no working
				[3]	

Question			Answer	Marks	Guidance
7	(i)			B1 B1	Tangent from where x = 0.5 does not meet x axis at positive x and explanation that the ln of a negative number cannot be found. Tangent from where x = 1 goes beyond second root and explanation that the value of the second root will be found. *NB. If tangents are not drawn then they must be specifically referred to in the explanation*
				[2]	
	(ii)		$\Rightarrow x_{r+1} = x_r - \dfrac{x_r^{\ 2} - \ln x_r - 2}{2x_r - \dfrac{1}{x_r}}$ $\Rightarrow x_{r+1} = x_r - \dfrac{x_r^{\ 3} - x_r \ln x_r - 2x_r}{2x_r^{\ 2} - 1}$ $\Rightarrow x_{r+1} = \dfrac{x_r\left(2x_r^{\ 2} - 1\right) - \left(x_r^{\ 3} - x_r \ln x_r - 2x_r\right)}{2x_r^{\ 2} - 1}$ $\Rightarrow x_{r+1} = \dfrac{x_r\left(2x_r^{\ 2} - 1\right) - x_r^{\ 3} + x_r \ln x_r + 2x_r}{2x_r^{\ 2} - 1} = \dfrac{x_r\left(x_r^{\ 2} + \ln x_r + 1\right)}{2x_r^{\ 2} - 1}$	M1 A1 A1	Application of correct formula- subscripts not required f ' seen Intermediate steps seen and all correct **AG**
				[3]	
	(iii)		x_2 = 0.131896(44...), x_3 = 0.137790(86...) $\Rightarrow \alpha$ = 0.13793	B1 B1	x_2 and x_3 to at least 6 dp answer.
				[2]	

Question			Answer	Marks	Guidance
8	(i)			B1 B1	Negative gradient for curve throughout with negative curvature and $y = x$ Clockwise iteration - at least three lines starting on curve
				[2]	
	(ii)		1.40000 1.37507 1.37945 1.37868 1.37882 α = 1.3788 1.37879 1.37880	B1 B1 B1	x_2 x_3 Value for α stated
				[3]	
	(iii)		1.40000 -0.02120 1.37507 0.00373 1.37945 -0.00065 -0.17422 1.37868 0.00012 -0.18132 Ratios –0.17 and –0.18 For curve: $y=(4-x)^{1/3} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{3}(4-x)^{-2/3}$ At $x=1.3788$, $\frac{\mathrm{d}y}{\mathrm{d}x} \approx -0.1753$ So ratios approximately equal to the gradient	B1 M1 B1	Ratios wrt diffn soi and value of gradient Comment
				[3]	

Question			Answer	Marks	Guidance
9	(i)		$\cos(-\theta)=\cos\theta$ or $\cos\theta$ is an even function or $\cos(2\pi-\theta)=\cos\theta$ or solving $\cos(2\alpha-\theta)=\cos\theta$ So $2\cos(-\theta)-1=2\cos\theta-1$ or giving $\alpha=0$	M1 A1	Using $\cos(2\pi-\theta)=\cos\theta$ earns A0 as outside domain
				[2]	
	(ii)			B1 B1 B1 B1	Loop drawn – ignore any extra part of the curve Point marked where $\theta=0$, (or r = 1) Tangents drawn All correct and symmetric with labelled tangents (Angles visibly at least at $\pi/4$) (Equation of tangents could be in cartesian form)
				[4]	
	(iii)		limits $\theta_1=-\frac{\pi}{3}$ and $\theta_2=\frac{\pi}{3}$ Area $=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,\mathrm{d}\theta=\frac{1}{2}\int_{\theta_1}^{\theta_2}(2\cos\theta-1)^2\,\mathrm{d}\theta$ $=\frac{1}{2}\int_{\theta_1}^{\theta_2}\left(4\cos^2\theta-4\cos\theta+1\right)\mathrm{d}\theta$ $=\frac{1}{2}\int_{\theta_1}^{\theta_2}\left(2\left(\cos 2\theta+1\right)-4\cos\theta+1\right)\mathrm{d}\theta$ $=\frac{1}{2}\left[\sin 2\theta+2\theta-4\sin\theta+\theta\right]_{\theta_1}^{\theta_2}$ $\Rightarrow$ Area $=\left[\sin 2\theta-4\sin\theta+3\theta\right]_0^{\pi/3}$ $=\left(\frac{\sqrt{3}}{2}-2\sqrt{3}+\pi\right)-0=\frac{2\pi-3\sqrt{3}}{2}$	B1 M1 M1 A1 A1	Limits Formula - need not have limits here but beware omission of $\frac{1}{2}$ due to integration being only half of area and is being doubled Using double angle Integrated function Answer does not need to be exact.
				[5]	

OCR (Oxford Cambridge and RSA Examinations)
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA

OCR Customer Contact Centre

Education and Learning
Telephone: 01223 553998
Facsimile: 01223 552627
Email: general.qualifications@ocr.org.uk

www.ocr.org.uk

For staff training purposes and as part of our quality assurance programme your call may be recorded or monitored

Oxford Cambridge and RSA Examinations
is a Company Limited by Guarantee
Registered in England
Registered Office; The Triangle Building, Shaftesbury Road, Cambridge, CB2 8EA
Registered Company Number: 3484466
OCR is an exempt Charity

OCR (Oxford Cambridge and RSA Examinations)
Head office
Telephone: 01223 552552
Facsimile: 01223 552553